A note on the type A_K real analytic vector fields *

Víctor Castellanos †
Universidad Juárez Autónoma de Tabasco, DACB
Abel Castorena ‡
Instituto de Matemáticas - UNAM, Campus Morelia
Manuel Cruz §
Facultad de Matemáticas, Universidad de Guanajuato

We prove in this paper that any real analytic vector field in $\mathfrak{X}(\mathbb{R}^n, k)$, $k = 2$ whose complex zero variety $Z_{\mathbb{C}}$ is a complete intersection of codimension 2 can be carried out to a type A_2 vector field.

En este artículo probamos que cualquier campo en $\mathfrak{X}(\mathbb{R}^n, k)$, $k = 2$ con una variedad $\mathbb{Z}_c$ de ceros complejos la cual es una intersección completa, puede ser llevado a un campo vectorial del tipo A_2.

Palabras clave: Caja de flujo, Campos vectoriales.
Keywords: Box flow, Vector fields.

1. Introduction

Let $X := \sum_{j=1}^{n} X^j \frac{\partial}{\partial x_j}$ be a germ of a real analytic vector field at $0 \in \mathbb{R}^n$ with an isolated singularity at 0. The coordinates of the vector field are elements in the ring of germs at 0 of real analytic functions in n variables $\mathcal{A}$. Recall that if $I \subset \mathcal{A}$ is an ideal, then $Z_{\mathbb{R}}(I) := \{\mathrm{x} \in \mathbb{R}^n : h(\mathrm{x}) = 0 \, \forall h \in I\}$ is the real zero variety associated to I, and similarly define $Z_{\mathbb{C}}(I)$ (see [2, 3]).

If $\mathfrak{X}(\mathbb{R}^n)$ denote the set which consists of all germs of real analytic vector fields at $0 \in \mathbb{R}^n$, we denote by $\mathfrak{X}(\mathbb{R}^n, k)$ the subset of $\mathfrak{X}(\mathbb{R}^n)$ which consists of elements that have a non algebraically isolated singularity at 0 and the complex zero variety $Z_{\mathbb{C}}(X)$ is a variety of codimension k.

In the case of codimension 1 and 2, this class of vector fields have been studied by Castellanos-Vargas in [1]. In that paper the author describes an algebraic formula for the index of a certain subclass of vector fields in $\mathfrak{X}(\mathbb{R}^n, k)$ which the author called *type* A_k.

*Recibido el 26 de abril de 2007 y aceptado el 14 de mayo de 2007

†**Address**: Carr. Cunduacán-Jalpa Km 1, Cunduacán Tabasco, México. A.P. 24 C.P. 86690. Tel.(+52)914 336-0928. **E-mail**: vicas@ujat.mx

‡**Address**: Apartado Postal 61-3 (Xangari), C. P. 58089, Morelia Mich. México **E-mail**: abel@matmor.unam.mx

§**Address**: Facultad de Matemáticas, UG. Jalisco s/n. Mineral de Valenciana, Guanajuato, Gto. 36240 **E-mail**: manuel@cimat.mx

2. Codimension k vector fields

According with [1], an element X of $\mathfrak{X}(\mathbb{R}^n, k)$ is called *type* A_k if:

1. X has the form
$$X = Y^1 \frac{\partial}{\partial x_1} + \cdots + Y^{k-1} \frac{\partial}{\partial x_{k-1}} + fY^k \frac{\partial}{\partial x_k} + \cdots + fY^n \frac{\partial}{\partial x_n}$$
where $f : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}, 0)$ and Y^j $(j = 1, \ldots, n)$ are germs of real analytic functions at 0.
2. The vector field $Y := Y^1 \frac{\partial}{\partial x_1} + \cdots + Y^n \frac{\partial}{\partial x_n}$ that we obtain from X after supressing the function f of the last $n - k + 1$ coordinates, is a vector field with an algebraically isolated singularity or, without singularities.
3. The complex algebraic variety $Z_{\mathbb{C}}$ is given by $Z_{\mathbb{C}}(X) = Z_{\mathbb{C}}(f, Y^1, \ldots, Y^{k-1})$.

Theorem 2.1. Let $X \in \mathfrak{X}(\mathbb{R}^n, k)$ such that $Z_{\mathbb{C}}(X)$ is a complete intersection algebraic variety defined by $\{f = 0\}$ and $\{Y^1 = \cdots = Y^{k-1} = 0\}$ and reduced outside zero. Then, there exist vector fields A and $B_1, \ldots, B_{k-1}$ such that
$$X = Af + B_1 Y^1 + \cdots + B_{k-1} Y^{k-1}.$$

Proof. Let $X = X^1 \frac{\partial}{\partial x_1} + \cdots + X^n \frac{\partial}{\partial x_n}$ be a real analytic vector field. Since $Z_{\mathbb{C}}(X) = V(f, Y^1, \ldots, Y^{k-1})$, the Nullstellensatz's theorem and the fact that $Z_{\mathbb{C}}(X)$ is reduced outside zero imply that $\sqrt{I_X} = (f, Y^1, \ldots, Y^{k-1})$. Since $I_X \subset \sqrt{I_X}$ we have that $X^j \in (f, Y^1, \ldots, Y^{k-1})$ and therefore, for each i, $i = 1, \ldots, n$ there exist real analytic functions a^i, $b^i_1, \ldots, b^i_{k-1}$ such that
$$X^i = a^i f + b^i_1 Y^1 + \cdots + b^i_{k-1} Y^{k-1}, \quad i = 1, \ldots, n. \tag{2.1}$$

Then we can define the vector fields
$$A(x) = \sum_{j=1}^{n} a^j \frac{\partial}{\partial x_j}, \quad B_1(x) = \sum_{j=1}^{n} b^j_1 \frac{\partial}{\partial x_j}, \quad \cdots, \quad B_{k-1}(x) = \sum_{j=1}^{n} b^j_{k-1} \frac{\partial}{\partial x_j}$$
which leads us to write $X = Af + B_1 Y^1 + \cdots + B_{k-1} Y^{k-1}$. □

Theorem 2.2. Let $X \in \mathfrak{X}(\mathbb{R}^n, k)$ as in the theorem 2.1 with $k = 2$. Then if A or B_1 in the decomposition of X as (2.1) does not vanish at zero, then there exists a local diffeomorphism ψ such that $\psi_* X$ is of the form A_2.

Proof. Let $k = 2$ be, and from 2.1, $B := B_1$, $g := Y^1$ and suppose that $B(0) \neq 0$. Then, by the Flow Box Theorem [4], there exists a diffeomorphism $\psi : U_0 \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $\psi_* B = \frac{\partial}{\partial x_n} = (0, \ldots, 1)$. Then,

$$
\begin{aligned}
\psi_* X &= \psi_*(Af + Bg) = \psi_*(Af) + \psi_*(Bg) \\
&= (\psi^* f)\psi_*(A) + (\psi^* g)\psi_* B \\
&= (\psi^* f)\psi_*(A) + (\psi^* g)\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} f'Y^1 \\ \vdots \\ f'Y^{n-1} \\ g' \end{pmatrix}.
\end{aligned}
$$

This means that $\psi_* X$ is a vector field of type A_2. □

The generalization of theorem 2.2 will be publish in further paper.

Referencias

[1] V. Castellanos-Vargas, *The Index of non algebraically isolated singularities*, Bol. Soc.Mat. Mexicana (3) Vol. 8, 2002.

[2] D. Cox, J. Little and D. O'Shea, *Ideals, Varieties and Algorithms*, UTM, Springer-Verlag, 1992.

[3] D. Eisenbud, *Commutative Algebra with a View toward Algebraic Geometry*, GTM 150, Springer-Verlag, 1996.

[4] J. Palis and W. deMelo, *Geometric Theory of Dynamical Systems*, Springer, 1982.