



DOCTORADO EN CIENCIAS MATEMÁTICAS
DIVISIÓN ACADÉMICA DE CIENCIAS BÁSICAS

**MODELACIÓN UNIDIMENSIONAL Y
BIDIMENSIONAL DE INUNDACIONES CON
FONDO MOVIBLE**

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TESIS DOCTORAL

DIRECTOR:

JUSTINO ALAVEZ RAMÍREZ

UNIVERSIDAD JUÁREZ AUTÓNOMA DE TABASCO, 2026



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TESIS

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PRESENTA:

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DIRECTOR:

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VILLAHERMOSA, TABASCO. SEPTIEMBRE 2026

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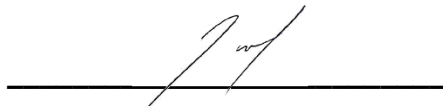


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ALAM YAIR HIDALGO DE LOS SANTOS



MARÍA FERNANDA JIMÉNEZ ALEGRÍA

Dedicatoria

Para Atilana y Oscar, quienes han creído en mí incondicionalmente, y han sido el motor de toda mi vida.

Para Erika y Regina, quienes hoy día son la motivación para seguir avanzando.

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México.

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Capítulo 1

Protocolo de tesis

1.1. Introducción

Hoy en día el cambio climático ha ocasionado que diversos fenómenos naturales tengan consecuencias muy costosas para la sociedad, como por ejemplo la DANA (*Depresión Aislada en Niveles Altos*) que afectó a la región de Sevilla en Octubre de 2024, inundaciones en la ciudad de México originadas por desbordes del gran canal, agosto 2025, las inundaciones que se suscitaron en Tabasco en los años 2000, 2007, 2020 por el desborde simultáneo de los ríos que atraviesan la planicie tabasqueña. Para mitigar el daño que éstos fenómenos puedan ocasionar es necesario estudiar su naturaleza y poder generar modelos que los predigan.

En este trabajo nos centraremos en la modelación matemática del flujo de agua en un canal o Río, y de la simulación numérica de dicho modelo matemático. La base para la resolución numérica de flujos no permanentes en superficie libre modelados por sistemas de leyes de conservación hiperbólicas está en la correcta solución analítica o aproximada del problema de Riemann asociado. En el caso del flujo de agua, el problema de Riemann se conoce como problema de *Dam-break* o *rotura de presa*, el cual se idealiza de la siguiente forma: **Existe una barrera vertical que divide un fluido (agua por ejemplo) en dos diferentes alturas**, h_l altura del agua a la izquierda y h_r altura del agua a la derecha, tal que $h_l < h_r$. Suponga que en un determinado instante de tiempo la barrera se quita (o se rompe) y el fluido fluye de izquierda a derecha. El problema se torna más complicado cuando se supone que el flujo del agua se da sobre un lecho movable, el cual a la izquierda de la barrera contenía una alta concentración de sedimento (arena, arcilla, lodo, grava) tal que el volumen de entrada del material puede ser considerable e incluso puede alcanzar, en ciertos momentos, el orden de la magnitud del volumen de agua contenido antes de quitar la barrera, como lo señalan Morichon, Desombre y Simian (2013) [P5]. En otras palabras, en este tipo de fenómenos se solucionan dos problemas de Riemann, el primero asociado al nivel del agua con h_l y h_r las alturas del agua a ambos lados de la presa y el segundo con b_l y b_r los niveles del fondo sedimentado a ambos lados de la presa. Tal como la altura del agua evoluciona,

lo harán los niveles del fondo sedimentado.

Es importante realizar una delimitación formal respecto al alcance de este trabajo. Si bien la visión original y el título del proyecto de investigación contemplaban proyecciones hacia la modelación bidimensional (2D) y su potencial implementación en herramientas de software hidroeinformático comercial/libre, el alcance efectivo, validado y presentado en esta tesis se centra en el análisis y simulación unidimensional. La trabajo matemático y computacional aquí desarrollada constituye una base sólida para futuras extensiones espaciales, las cuales serán objeto de trabajo futuro.

1.2. Marco teórico

Muchos modelos sobre rotura de presa pueden ser encontrados en la literatura, los cuales son usados para predecir el flujo y la extensión de inundaciones, pero la mayoría de estos modelos son solo aplicables para fondo fijo. En eventos extraordinarios, como por ejemplo, rotura de presas o fallas de diques, el flujo induce movimientos severos de sedimentos en varias formas, ya sea, flujo de partículas (material), flujo lodoso, corrientes cargadas de sedimentos con rocas o lodo, como se menciona en Xia, et al (2010) [P10]. En general existen fuertes interacciones entre el flujo, el transporte de sedimentos y la morfología del canal. Experimentos en laboratorio han contribuido significativamente a entender la hidráulica de una rotura de presa sobre fondo movable, Ji Li, et al (2013) [P6]. Los cambios rápidos del fondo inducen un salto de la superficie del agua mientras que un escurrimiento aparece en el punto dónde estaba ubicada la presa, como se expresa por Morinchon, Desombre y Simian (2013) [P5].

Dado que los experimentos físicos requieren de mucho tiempo para su preparación y de un coste monetario significativo, una forma alternativa de mejorar el entendimiento del flujo de una rotura de presa con fondo movable, **flujo del transporte de sedimento**, es mediante la modelación computacional de los mismos, en ese campo el trabajo de Capart y Young (1998) [P2] ha actuado como un catalizador para el desarrollo de muchos métodos numéricos.

Para poder llevar a cabo la modelación computacional del flujo del transporte de sedimento es necesario tener un sistema de ecuaciones diferenciales que tome en cuenta las características principales del fenómeno, como lo son, los procesos de cambios morfológicos en el fondo y la evolución de la altura del fluido y su velocidad. Una opción para poder hacer esto es, utilizar modelos que por una parte predigan el comportamiento del fluido y después el cambio del fondo (modelos desacoplados) o modelos que calculen todos los procesos a la vez (modelos acoplados). Los primeros métodos numéricos que simulaban el flujo de una rotura de presa sobre fondo movable adoptaron soluciones desacopladas, las cuales no toman en cuenta los efectos del transporte del sedimento y la deformación del fondo debida al movimiento del flujo.

Cao et al (2004) [P4], presentan un modelo en el cual el transporte de sedimentos inducido por el flujo del agua y la evolución morfológica se tiene en cuenta mediante una ley de transporte de fondo. En dicho trabajo se presenta una caracterización de las

soluciones del problema de Riemann asociado al flujo del transporte de sedimentos. Wu y Wang (2007) [P9], proponen un modelo unidimensional acoplado para investigar los procesos morfodinámicos ocasionados por el flujo de una rotura de presa sobre lecho móvil.

Modelos bidimensionales para los flujos de transporte de sedimentos con fondo móvil que calculen simultáneamente la hidrodinámica y la morfología del lecho son reportados esporádicamente dada la complejidad del flujo del transporte de sedimento y la evolución del fondo. Simpson y Castelltort (2006) [P8], han extendido los modelos unidimensionales con soluciones acopladas a un modelo 2D para el flujo de lámina libre, para solucionar numericamente este tipo de problemas proponen utilizar el método de Godunov de primer orden, cuya idea es suponer que se puede resolver el problema de Riemann asociado de manera exacta para un tiempo corto, y con esto construir flujos numéricos que ayuden a calcular la evolución temporal del fenómeno de estudio.

J. Xia et al (2010) [P10], acoplan un modelo 2D hidrodinámico con un módulo de transporte de sedimentos y evolución del fondo para predecir simultáneamente la hidrodinámica, concentración de sedimentos y cambio morfológicos. Utilizan este modelo para investigar, la influencia de las diferentes composiciones del material del fondo en los niveles de inundaciones y los cambios morfológicos del fondo.

Una forma atractiva para modelar una rotura de presa son los modelos de dos fases o dos capas, en la primera capa se tendría el agua limpia y en una segunda capa se consideran las interacciones explícitas entre el sedimento y el agua debido a la estructura estratificada del flujo. Algunos modelos doble capa DL (*Double-Layer*) son simplificados, puesto que se supone constante la concentración de carga del fondo, la cual en un principio es una incógnita a solucionar por el método numérico. En los modelos existentes SDL (*Simplified Double Layer*) este valor se especifica a priori, introduciendo incertidumbre y con ello violando la ley de conservación de masa del sedimento, Ji Li et al (2013) [P6].

Un modelo multi-capas libre de malla fue propuesto por Shakibaenia y Jim (2011) [P7], en este trabajo demuestran que es necesario un modelo no newtoniano para reproducir detalladamente las características del flujo y del sedimento.

Ji Li et al (2013) [P6], construyen un sistema de leyes de conservación de masa y momento para cada capa con las relaciones que representan las interfaces, la profundidad y las tensiones tangenciales. Las ecuaciones que gobiernan cada capa son un sistema cuasi hiperbólico no-homogéneos mientras que la interacción entre las capas y el fondo móvil, la resistencia de las fronteras y la acción gravitacional son representadas en el término fuente. Para cada sistema hiperbólico se calcula la solución usando métodos flux-splitting y un esquema Runge-kutta para solucionar la ecuación diferencial ordinaria que induce el término fuente.

Los modelos existentes reportan buenos resultados, pero al momento de ser contrastados con la realidad los resultados muestran diferencias significativas tanto cualitativas como cuantitativas

1.3. Justificación

El problema de las inundaciones es un fenómeno que está afectando cada vez a más personas, ya sea por el crecimiento de la mancha urbana que lleva a construir en áreas que son vasos reguladores, o por las intensas lluvias que se dan por el cambio climático, las cuales ocasionan el desborde de los ríos. Tradicionalmente el análisis de las inundaciones se lleva a cabo considerando solamente las variables hidrodinámicas y desprecian tanto la concentración de sedimentos y los procesos de erosión y depósito o éstos últimos son tomados en cuenta de forma desacoplada o simplificada. Trabajar de esta forma para predecir el alcance y magnitud de inundaciones no es ideal ya que los cambios morfológicos afectan al flujo, pueden llegar a ocasionar mayores niveles de inundación y ocasionan que la velocidad del flujo se incremente. En eventos de inundaciones se pueden originar fallas en estructuras (puentes, carreteras, edificios) debido a socavones, los cuales en modelos unidimensionales o bidimensionales no pueden ser predecidos de manera correcta. Tener un modelo que tenga en cuenta todos los fenómenos que ocurren durante las inundaciones y una forma adecuada de poder solucionarlo es de suma importancia para predecir de forma precisa el nivel de una inundación, hasta dónde llegará, qué velocidades alcanzará en determinadas áreas, qué zonas podrían tener comprometidas sus estructura. El poder predecir inundaciones es de vital importancia para la toma de decisiones oportunas que ayuden a mitigar los daños de éstas y las pérdidas (humanas y económicas) que se origina como resultados de estos eventos extraordinarios en la naturaleza.

1.4. Pregunta de investigación

¿De qué manera la implementación de esquemas de volúmenes finitos para resolver sistema acoplados de 1D para el transporte de sedimentos permite reproducir consistentemente la evolución morfodinámica en este tipo de flujos?

1.5. Hipótesis

Los modelos en ecuaciones diferenciales parciales unidimensionales acoplados, resueltos a través de esquemas en volúmenes finitos bien equilibrados, capturan con mayor precisión la evolución del lecho móvil al integrar de forma simultánea la interacción entre los procesos hidrodinámicos y los procesos morfodinámicos del flujo.

1.6. Objetivo general

Desarrollar y analizar un modelo matemático unidimensional en ecuaciones diferenciales parciales para el transporte del sedimento cuya solución esté basada en esquemas

de volúmenes finitos que permita calcular la hidrodinámica, el transporte de sedimentos y los cambios morfológicos en flujos de superficie libre

1.6.1. Objetivos específicos

1. Analizar la formulación matemática y las propiedades del sistema acoplado unidimensional de las ecuaciones de Saint-Venant y Exner.
2. Formular e implementar esquemas numéricos de volúmenes finitos adaptados para resolver adecuadamente los términos fuente no conservativos en 1D.
3. Recopilar y seleccionar datos de pruebas experimentales de laboratorio reportados en la literatura (como roturas de presa sobre fondo móvil y evolución de canales).
4. Validar la precisión, estabilidad y capacidad de conservación de los esquemas numéricos mediante la simulación de casos de estudio morfodinámicos unidimensionales.
5. Comparar el desempeño computacional y la resolución de las discontinuidades entre los diferentes esquemas evaluados sobre la evolución del perfil longitudinal del fondo.

1.7. Metodología

1. Se analizó la estructura matemática del sistema de ecuaciones acopladas de Saint-Venant–Exner en una dimensión espacial.
2. Se formuló una discretización del sistema mediante el método de volúmenes finitos, integrando *approximated Riemann solver*, tales como los Q-esquemas, Q-esquemas con términos no conservativos, con un tratamiento consistente de los términos fuente.
3. Se implementaron los algoritmos computacionales correspondientes a los esquemas numéricos evaluados para la simulación de la hidrodinámica y la evolución del fondo en el caso unidimensional.
4. Se recopiló y procesó datos experimentales de laboratorio reportados en la bibliografía para casos de prueba de referencia, como rotura de presa (*Dam break problem*) sobre fondo móvil y evolución de canales.
5. Se ejecutaron las simulaciones numéricas unidimensionales sobre los casos de prueba seleccionados, contrastando los resultados obtenidos contra las mediciones experimentales.

6. Se realizó un análisis comparativo del desempeño de los esquemas numéricos en cuanto a estabilidad, resolución de discontinuidades y reproducción de la morfodinámica del flujo.

1.8. Cronograma de actividades

Las actividades a realizarse en este trabajo se planean conforme a la calendarización propuesta en la tabla 1.1. Se propone desarrollar ciertas actividades durante el transcurso de cada semestre, la cuales pueden ser alteradas de acuerdo a las necesidades que se vayan presentando, pero se compromete la entrega de los siguientes productos al final de cada semestre.

Actividad	Periodo semestral
Revisión bibliográfica de los modelos tipo Saint-Venant–Exner unidimensionales y análisis teórico de los métodos de volúmenes finitos tipo Riemann solver.	Segundo semestre.
Formulación matemática de la discretización por volúmenes finitos 1D con tratamiento de términos fuente no conservativos y programación de los esquemas numéricos.	Tercer semestre.
Recopilación de datos de pruebas experimentales de laboratorio reportados en la literatura (rotura de presa y evolución de canales) y ejecución de simulaciones numéricas unidimensionales.	Cuarto semestre.
Validación y análisis comparativo del desempeño de los esquemas numéricos (estabilidad, conservación y resolución de discontinuidades). Someter un artículo para ser publicado en revista de alto impacto.	Quinto semestre.
Entrega del documento final para revisión por parte del comité sinodal y preparación de la defensa.	Sexto semestre.
Redacción del trabajo recepcional (tesis doctoral).	Durante el tiempo total del proyecto.

Tabla 1.1: Cronograma de Actividades.

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Capítulo 2

Artículo científico publicado en revista especializada

En este capítulo se presentan los resultados publicados en el artículo *Numerical simulation of bed load and suspended load sediment transport using well balanced numerical schemes*, el cual puede ser consultado en el enlace <https://doi.org/10.1007/s42967-021-00162-1>.

En este trabajo se estudia un sistema de ecuaciones hiperbólicas que modela el flujo del transporte de sedimento, tanto por carga de fondo como por carga en suspensión. El transporte por carga de fondo se hace siguiendo la formulación de Grass.

En principio, se presenta el modelo matemático que se obtendría sin suponer que la concentración de la mezcla de agua y sedimento es constante, pero para el tratamiento numérico del modelo matemático se utiliza dicha suposición, esto para tener en cuenta los efectos de la densidad de la concentración solo en el término fuente, es decir, para tratar el modelo como un sistema de leyes de balance y poder con ello primero resolver el sistema conservativo asociado al modelo y con dicha solución resolver la ecuación diferencial ordinaria que induce el término fuente.

Se desarrollan tres esquemas numéricos de volúmenes finitos para la solución del sistema. El primer esquema se basa en el método Q-esquema de VanLeer, en el cual, el flujo numérico se define utilizando la eigen-estructura del sistema, porque se conocen las formas explícitas de los eigenvalores e ingenvectores. El segundo esquema se basa en la idea de definir el flujo numérico tomando en cuenta la aparición de términos no conservativos, esto ayuda a que en la eigen-estructura del sistema aparezca la información que da la concentración de la carga del sedimento. El último método calcula el flujo numérico utilizando el flujo numérico tipo HLLC-S, es decir, un método que toma en cuenta los saltos en la onda rápida y lenta del sistema, así como en la discontinuidad de contacto y más aún, considera el salto que se da debido al término fuente que contiene los procesos de erosión y depósito, es decir, toma en cuenta las soluciones de los problemas de Riemann en cada una de las ondas que pueden aparecer en el sistema.

En este trabajo, quien presenta la tesis, Juan Carlos González Aguirre, he colaborado en el desarrollo del estado del arte del trabajo, he trabajado en el análisis de los modelos

matemático, el cálculo de las velocidades características y los valores propios de ambas versiones del modelo, así como la implementación de los códigos que desarrollan los esquemas de volúmenes finitos Q-esquema y QNC, así como la definición del esquema descentrado para el cálculo del flujo del transporte de sedimentos. En el artículo fungo como primer autor y como autor por correspondencia.

2.1. Introduction

The mathematical modelling of sediment transport has become an important subject as extreme hydrological events intensified and increased in numbers, as a result of climate change, see [8]. According to [24], the mathematical models proposed can be classified as fully coupled models (FCM), partially coupled models (PCM) and decoupled models (DM). The main difference between them is the manner in which they deal with the interaction between water flux, sediment transport and bed evolution. The FCM take into account the hydrodynamic and both the suspended load and the bedload transport. In these models, the bed evolution is modelled using the Exner equation (see [16, 17]) with source terms. The PCM consider the hydrodynamic and suspended load sediment transport. In this kind of model, the bedload transport is not taken into account for modelling the bed evolution. The DM compute the sediment transport using the shallow water equations coupled with the bed conservation equation. In this kind of model the suspended load sediment transport is neglected.

Another difference between the models is the way each of them considers the interaction between the flux, sediment transport and the morphological evolution inside the structure of the mass conservation equation and the momentum conservation equation. The FCM can have up to two additional terms on the right-hand side of the mass conservation equation. These terms quantify the rate of water entrainment (e.g. rainfall, infiltration, etc.) and the bed deformation. On the right-hand side of momentum conservation equation two additional terms are added, which measure the effects of variable sediment concentration and momentum transfer due to sediment exchange between the water and the movable bed ([8]). Examples of FCM can be found in [8, 24, 28, 3]. These models consider hydrostatic pressure. Unlike them, in [7] a non-hydrostatic system is used, in this kind of approach, special focus must be put due to appearance of numerical instabilities which are given by the non-hydrostatic distribution.

The PCM do not consider these additional terms in the shallow water equations because the sediment conservation and bed conservation are computed with the information provided by the hydrodynamics of clear water. Therefore, the interaction that may exist between the flux and sediment is ignored for the hydrodynamic equations. Examples of this kind of model can be found in [9] and [30]. In order to compute the morphological evolution, the majority of PCM does not take into account the bedload transport, although in works such as [37, 30], it is included.

The DM use the shallow water equations coupled with the bed conservation equation for modelling the sediment transport flux. The bed conservation, in this kind of model,

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ORIGINAL PAPER



Numerical Simulation of Bed Load and Suspended Load Sediment Transport Using Well-Balanced Numerical Schemes

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Abstract

Sediment transport can be modelled using hydrodynamic models based on shallow water equations coupled with the sediment concentration conservation equation and the bed conservation equation. The complete system of equations is made up of the energy balance law and the Exner equations. The numerical solution for this complete system is done in a segregated manner. First, the hyperbolic part of the system of balance laws is solved using a finite volume scheme. Three ways to compute the numerical flux have been considered, the Q -scheme of van Leer, the HLLCS approximate Riemann solver, and the last one takes into account the presence of non-conservative products in the model. The discretisation of the source terms is carried out according to the numerical flux chosen. In the second stage, the bed conservation equation is solved by using the approximation computed for the system

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Figura 2.1: Primera página del artículo de investigación

presents a convective term that quantifies the bedload transport. Examples of these models have been used in [1, 10, 32, 25]. A summary of these different kinds of models can be seen in Table 2.1.

The first equation to model the bed balance was proposed by Felix Exner in his river morphology studies, [16, 17]. In this equation, the sediment flux was approximated through flux velocity. Currently in the literature different formulations are used to compute the bedload flux, for instance [31, 14, 21, 41].

The majority of the formulations proposed considers the critical Shields parameter, [36], to control the bed movement, nevertheless, the Grass law for bedload transport [21] does not consider this parameter, bed movement starts when the flux velocity is different to zero. The Grass law also allows to measure if the interaction between bed and flux is strong or weak. In [10], a formulation where some bedload transport models are rewritten in analogous form to the law of Grass, is proposed.

In order to solve numerically the sediment transport problem, several numerical strategies have been proposed. The majority of them is based on the finite volume numerical schemes, other strategies are based on cellular automata, although the use of cellular automata for modelling the sediment flux is a field that have been few explored, such as is pointed in [24]. As well, recently, the finite elements framework has been employed to compute the numerical solution of the sediment transport on shallow water equations, for instance [15, 2, 18] have employed this technique. Up to our knowledge the extension of approximate Riemann solvers is the most suitable option for the numerical resolution of the sediment transport, some of them are exposed in [39] and [26].

In order to solve the complete system for sediment transport modelling, there are two means found in the literature, regardless of whether the model is FCM, PCM or DM. In the first one, the hyperbolic system of balance laws is solved through an approximate Riemann solver scheme. Then, the bed conservation equation is solved by using the information provided by this scheme. This method has been used in works like [8, 24, 32, 25]. In [13] is discussed that this way to solve the sediment transport can fail due to the characteristic velocity related to the bedload is not taken into account to solve the hydrodynamics of water sediment mixture. The second method computes the solution in coupled form, as in [28, 3, 29, 30, 34]. In order to compute the depth of bed together with the other variable the bed conservation equation is rewritten introducing a new conservative variable, which depends on the depth of bed and the suspended load.

According to Cao [8], the water flux, the sediment transport and the morphological evolution of the bed are strongly coupled. The rate of deformation can be significant in comparison with the flux evolution and the use of a fully coupled model that takes into account all conservation laws is needed. In this paper, a FCM is considered, which takes into account both the suspended load sediment transport and the bedload sediment transport. The structure of this work is as follows: in section 2.2 the mathematical model is presented and the employed formulations for erosion and deposition are described. In

section 2.3 a general compact formulation will be presented, as well as an alternative way to write the model considering the presence of non conservative product ([10, 11]) and the study of the hyperbolicity of the model. The numerical schemes to be used will be shown in section 2.4, as well as the steady stationary solution which we want to preserve. Finally, in section 2.5 we validate the mathematical model and the numerical schemes by carrying out several numerical experiments and making comparison between the obtained numerical results and the experimental data.

2.2. Mathematical model

In this work, one-dimensional shallow water flow, over an erodible bottom composed of uniform, non-cohesive sediment, is analysed. This phenomenon is depicted in Figure 2.2.

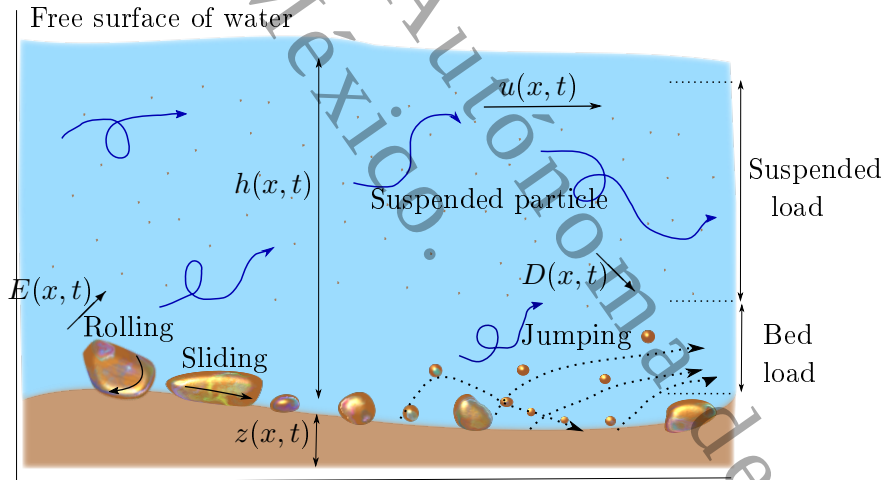


Figura 2.2: Sketch of bedload and suspended-load sediment transport.

The dynamics of the movable bottom as a result of hydrodynamic behaviour can be modelled by using the system of balance laws made up by the mass conservation equation, the momentum conservation equation, the sediment conservation equation and the bed conservation equation, namely,

$$\frac{\partial h}{\partial t}(x, t) + \frac{\partial hu}{\partial x}(x, t) = \frac{E(x, t) - D(x, t)}{1 - p}, \quad (2.1)$$

$$\begin{aligned} \frac{\partial(hu)}{\partial t}(x, t) + \frac{\partial(hu^2 + \frac{1}{2}gh^2)}{\partial x}(x, t) = & gh(x, t) \left(-\frac{\partial z}{\partial x}(x, t) - S_f(x, t) \right) \\ & - \delta \frac{(\rho_s - \rho_w)gh^2(x, t)}{2\rho(x, t)} \frac{\partial c}{\partial x}(x, t) \\ & - \frac{(\rho_0 - \rho(x, t))(E(x, t) - D(x, t))u(x, t)}{\rho(x, t)(1 - p)}, \end{aligned} \quad (2.2)$$

$$\frac{\partial(hc)}{\partial t}(x, t) + \frac{\partial(huc)}{\partial x}(x, t) = E(x, t) - D(x, t), \quad (2.3)$$

$$\frac{\partial z}{\partial t}(x, t) + \frac{1}{1 - p} \frac{\partial q_b}{\partial x}(x, t) = \frac{D(x, t) - E(x, t)}{1 - p}, \quad (2.4)$$

where

- $h(x, t)$ is the depth of water (m),
- $u(x, t)$ is the averaged-flux velocity (m/s),
- $c(x, t)$ is the averaged-flux volumetric sediment concentration (dimensionless),
- $z(x, t)$ is the depth of bed (m),
- $\rho(x, t)$ is the density of water-sediment mixture (kg/m³),
- $S_f(x, t)$ is the friction slope computed by using equation (2.8),
- g is the gravitational acceleration (m/s²),
- p is the bed sediment porosity,
- $E(x, t)$ is the sediment entrainment (m/s) (see [8]),
- $D(x, t)$ is the sediment deposition (m/s) (see [8]),
- ρ_w is the water density (kg/m³),
- ρ_s is the sediment density (kg/m³),
- ρ_0 is the saturated bed density (kg/m³),
- $q_b(x, t)$ is the bed-load discharge (m²/s) (see [21]).
- δ is defined as

$$\delta = \begin{cases} 1 & \text{if } \theta \geq \theta_c, \\ 0 & \text{if } \theta < \theta_c, \end{cases} \quad (2.5)$$

where θ and θ_c are the Shields parameter and Critical Shields value, respectively. These values are defined in section 2.2.1.

The right-hand side of the mass conservation law for the water sediment mixture (2.1), is significant for the processes where the sediment transport and the morphological evolution are active (see [8]). The second term on the right-hand side of the momentum conservation equation (2.2), measures the effects of the variable sediment concentration on the flux direction. In [30] is pointed out that non-uniform sediment concentration over irregular bottom in quiescent water may cause perturbation of water at rest state, in order to deal with this issue, the second term on the right hand side is taken into account from the moment that the flux can move the bottom, that is to say, when the Shields parameter is greater than critical Shields parameter. The third term represents the momentum transfer due to the exchange between the water and the erodible bed, [8]. Equation (2.3) is the suspended sediment conservation equation. The morphodynamics are well-modelled by the Exner equation (2.4), which states that the time variation of the sediment layer, in a certain volume, is due to the net variation of the solid transport through the boundaries of the volume, [16, 17]. The formulation of the bed load discharge $q_b(x, t)$ can be based on empirical laws, some of the most often used were proposed in [31], [21] and [41]. In order to compute the bed load discharge, the law of Grass [21] was chosen for this work

$$q_b(x, t) = A_g u^3(x, t), \quad (2.6)$$

where A_g is a constant value that takes into account the diameter of the particle and the kinematic viscosity, usually it is obtained experimentally. The values for A_g range from zero to one, so that if A_g is close to zero, then the model shows a weak interaction between the sediment bottom and the fluid. On the other hand, if A_g is close to one, then the interaction between the sediment bottom and the fluid is strong, [10].

2.2.1. Morphological conditions

The density of the water-sediment mixture, ρ , and the density of the saturated bed, ρ_0 , are computed as

$$\rho(x, t) = \rho_w(1 - c(x, t)) + \rho_s c(x, t) \quad \text{and} \quad \rho_0 = \rho_w p + \rho_s(1 - p). \quad (2.7)$$

The friction term is computed using the following formula

$$S_f(x, t) = c_f u(x, t) |u(x, t)|, \quad (2.8)$$

where c_f is computed as

- $c_f = \frac{\eta^2}{h(x, t)^{\frac{4}{3}}}$, being η the dimensionless Manning coefficient.
- c_f is the friction coefficient.

The sediment exchange between the bed and the water column is determined through the sediment entrainment due to turbulence, $E(x, t)$, and the sediment deposition due to gravitational action, $D(x, t)$, (see [8]). The sediment entrainment is given by

$$E(x, t) = \begin{cases} \varphi \frac{(\theta(x, t) - \theta_c)u(x, t)}{h(x, t)d^{0.2}} & \text{if } \theta(x, t) \geq \theta_c, \\ 0 & \text{else,} \end{cases} \quad (2.9)$$

the sediment deposition reads

$$D(x, t) = \alpha c(x, t)\omega_0(1 - \alpha c(x, t))^m, \quad (2.10)$$

where:

- φ is a constant dimensionless value determined by

$$\varphi = \phi \frac{560(1 - p)\nu^{0.8}}{3(sg)^{0.4}\theta_c}. \quad (2.11)$$

This expression was deducted following [8].

- ϕ is a dimensionless value that depends on the phenomenon to recreate,
 - ν is the kinematic viscosity of the fluid (m^2/s),
 - $s = \frac{\rho_s}{\rho_w} - 1$ is the dimensionless submerged specific gravity of the sediment,
 - θ_c is the dimensionless critical Shields parameter which indicates the start of sediment movement.
- The dimensionless Shields parameter or mobility parameter is computed using the formula

$$\theta(x, t) = \frac{u_*^2(x, t)}{sgd}, \quad (2.12)$$

where

- d is the diameter of the particle (10^{-3}m),
- $u_*(x, t)$ is the friction velocity (m/s), that is given by the relation

$$u_*(x, t) = \sqrt{\frac{\tau(x, t)}{\rho}},$$

- $\tau(x, t)$ is the threshold stress of bottom computed using the following expression

$$\tau = \frac{g\rho\eta^2u(x, t)|u(x, t)|}{h(x, t)^{1/3}}.$$

- The drop velocity ω_0 (m/s) is computed using the formulation proposed in [43]

$$\omega_0 = \sqrt{\left(\frac{13.95\nu}{d}\right)^2 + 1.09sgd} - \frac{13.95\nu}{d}, \quad (2.13)$$

- m is an exponent computed by using the Reynolds number of the particle, see [27].

$$m = 4.45R^{-0.1} \quad \text{where,} \quad R = \frac{\omega_0 d}{\nu}. \quad (2.14)$$

- α is the nonequilibrium adaptation coefficient of suspended load, [42]. Following [8], this coefficient is defined as $\alpha = \min\{2, (1-p)/c\}$. This relation holds that near-bed supend-load concentration αc must be smaller than the bed material concentration $1-p$, [42].

In Table 2.1 the different kinds of models which have been discussed in section 2.1 are shown, where the one dimensional shallow water equations are given by

$$\frac{\partial h}{\partial t}(x, t) + \frac{\partial hu}{\partial x}(x, t) = 0, \quad (2.15)$$

$$\frac{\partial hu}{\partial t}(x, t) + \frac{\partial hu^2 + \frac{1}{2}gh^2}{\partial x}(x, t) = -gh \frac{\partial z}{\partial x}(x, t). \quad (2.16)$$

Tabla 2.1: Models summary.

Model	Equations	Remarks
FCM	(2.1), (2.2), (2.3), (2.4)	Complete model used in this work
PCM	(2.15), (2.16), (2.3), (2.4)	Bedload discharge is neglected
DM	(2.15), (2.16), (2.4)	RHS of (2.4) is neglected

2.3. Problem statement

In order to write the mathematical model (2.1)-(2.3) in a compact manner the following notation is employed

- Conservative variables,
 - $w_1 = h(x, t)$ is the depth of water (m),
 - $w_2 = h(x, t)u(x, t)$ is the unit discharge (m²/s),
 - $w_3 = h(x, t)c(x, t)$ is the suspended sediment concentration (kg/m²),

- $\mathbf{W}(x, t) := (w_1, w_2, w_3)^t$ is the vector of conservative variables.

- Physical flux,

$$\mathbf{F}(\mathbf{W}) = \begin{pmatrix} w_2 \\ \frac{w_2^2}{w_1} + \frac{1}{2}gw_1^2 \\ \frac{w_2w_3}{w_1} \end{pmatrix}. \quad (2.17)$$

- Source term,

$$\begin{aligned} \mathbf{S}(x, t, \mathbf{W}) &= \begin{pmatrix} \frac{E - D}{1 - p} \\ -gw_1 \left(\frac{\partial z}{\partial x} + S_f \right) - \frac{(\rho_s - \rho_w)gw_1^2}{2\rho} \frac{\partial}{\partial x} \left(\frac{w_3}{w_1} \right) - \frac{\rho_0 - \rho}{\rho} \frac{E - D}{1 - p} \frac{w_2}{w_1} \\ E - D \end{pmatrix}. \end{aligned} \quad (2.18)$$

The source term, $\mathbf{S}(x, t, \mathbf{W})$, is split into four parts: bed slope, $\mathbf{S}_1(x, t, \mathbf{W})$, friction slope, $\mathbf{S}_2(x, t, \mathbf{W})$, sediment entrainment and sediment deposition process, $\mathbf{S}_3(x, t, \mathbf{W})$, and volumetric sediment concentration slope, $\mathbf{S}_4(x, t, \mathbf{W})$, so that:

$$\mathbf{S}(x, t, \mathbf{W}) = \sum_{k=1}^4 \mathbf{S}_k(x, t, \mathbf{W}),$$

where

$$\mathbf{S}_1(x, t, \mathbf{W}) = \begin{pmatrix} 0 \\ -gw_1 \frac{\partial z}{\partial x}(x, t) \\ 0 \end{pmatrix}, \quad (2.19)$$

$$\mathbf{S}_2(x, t, \mathbf{W}) = \begin{pmatrix} 0 \\ -gw_1 S_f(x, t) \\ 0 \end{pmatrix}, \quad (2.20)$$

$$\mathbf{S}_3(x, t, \mathbf{W}) = \begin{pmatrix} \frac{E(x, t) - D(x, t)}{1 - p} \\ -\frac{\rho_0 - \rho(x, t)}{\rho(x, t)} \frac{E(x, t) - D(x, t)}{1 - p} \frac{w_2}{w_1} \\ E(x, t) - D(x, t) \end{pmatrix}, \quad (2.21)$$

and

$$\mathbf{S}_4(x, t, \mathbf{W}) = \begin{pmatrix} 0 \\ -gw_1^2 \frac{(\rho_s - \rho_w)}{2\rho(x, t)} \frac{\partial}{\partial x} \left(\frac{w_3}{w_1} \right) \\ 0 \end{pmatrix}. \quad (2.22)$$

Using this notation, the equations (2.1)-(2.3) can be rewritten in vectorial form as

$$\frac{\partial \mathbf{W}}{\partial t}(x, t) + \frac{\partial \mathbf{F}(\mathbf{W})}{\partial x}(x, t) = \mathbf{S}(x, t, \mathbf{W}), \quad (2.23)$$

for $x \in [0, \mathcal{L}]$ and $t \in [0, T]$, where $\mathcal{L}$ is the length of the considered domain channel.

- Initial conditions:

$$\mathbf{W}(x, 0) = \mathbf{W}_0(x), \quad \text{and} \quad z(x, 0) = z_0(x), \quad x \in [0, \mathcal{L}].$$

In practice, the initial values for averaged depth of water $h_0(x)$, averaged flux velocity $u_0(x)$, averaged flux volumetric sediment concentration $c_0(x)$ and depth of bed $z_0(x)$, are known.

- Boundary conditions:

Conditions at the left end of the channel are detailed and the conditions of the right end are similar.

- Inflow ($w_2(0, t) > 0$): $w_2(0, t) = q_l(t)$,
- Outflow ($w_2(0, t) < 0$): $w_2(0, t) = q_l(t)$,
- Wall $w_2(0, t) = 0$,
- Either inflow or outflow or wall: $w_1(0, t) = h_l(t)$ and $w_3(0, t) = h_l(t)c_l(t)$,

where $q_l(t)$, $h_l(t)$, $h_l(t)c_l(t)$ are the unit discharge, the depth of water and the suspended sediment concentration at time t , respectively, at the left end of the domain.

2.3.1. Hyperbolic character of the system

The balance law (2.23) can be written in quasi-linear form as

$$\frac{\partial \mathbf{W}}{\partial t}(x, t) + \mathbf{A}(\mathbf{W}(x, t)) \frac{\partial \mathbf{W}}{\partial x}(x, t) = \mathbf{S}(x, t, \mathbf{W}), \quad (2.24)$$

where $\mathbf{A}(\mathbf{W}(x, t))$ is the Jacobian matrix of the physical flux $\mathbf{F}(\mathbf{W}(x, t))$, which is given by

$$\mathbf{A}(\mathbf{W}(x, t)) = \begin{pmatrix} 0 & 1 & 0 \\ gw_1 - \frac{w_2^2}{w_1^2} & 2\frac{w_2}{w_1} & 0 \\ -\frac{w_2 w_3}{w_1^2} & \frac{w_3}{w_1} & \frac{w_2}{w_1} \end{pmatrix}. \quad (2.25)$$

The eigenvalues of this matrix written as function of physical variables take the form:

$$\lambda_1 = u - \sqrt{gh}, \quad \lambda_2 = u, \quad \lambda_3 = u + \sqrt{gh},$$

and the corresponding eigenvectors are:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ \lambda_1 \\ c \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ \lambda_3 \\ c \end{pmatrix}.$$

2.3.2. Non conservative system

As the derivative of the conservative variables appears inside the source term $\mathbf{S}_4(x, t\mathbf{W},)$, this source term can be seen as a non conservative product, [10]. Taken into account this fact, the hyperbolic system of conservation laws can be rewritten as

$$\frac{\partial \mathbf{W}}{\partial t} + \mathbf{A}(\mathbf{W}) \frac{\partial \mathbf{W}}{\partial x} + \mathbf{B}(\mathbf{W}) \frac{\partial \mathbf{W}}{\partial x} = \sum_{k=1}^3 \mathbf{S}_k(x, t, \mathbf{W}). \quad (2.26)$$

The effects of the variable sediment concentration are taken into account in the third term on the left-hand side of (2.26), also called the non conservative product. The matrix $\mathbf{B}(\mathbf{W})$ takes the form

$$\mathbf{B}(\mathbf{W}) = \begin{pmatrix} 0 & 0 & 0 \\ -gw_1 \frac{\rho - \rho_w}{2\rho} & 0 & gw_1 \frac{\rho_s - \rho_w}{2\rho} \\ 0 & 0 & 0 \end{pmatrix}. \quad (2.27)$$

Quasi linear form and hyperbolic character

The system of balance laws (2.26) can be written in quasi linear form as

$$\frac{\partial \mathbf{W}}{\partial t} + \mathcal{A}(\mathbf{W}) \frac{\partial \mathbf{W}}{\partial x} = \sum_{k=1}^3 \mathbf{S}_k(x, t, \mathbf{W}), \quad (2.28)$$

where

$$\mathcal{A}(\mathbf{W}) = \mathbf{A}(\mathbf{W}) + \mathbf{B}(\mathbf{W}) \quad (2.29)$$

and its eigenvalues are given by

$$\tilde{\lambda}_1 = u - \sqrt{gh}; \quad \tilde{\lambda}_2 = u; \quad \tilde{\lambda}_3 = u + \sqrt{gh},$$

and the associated eigenvectors read

$$\tilde{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ \tilde{\lambda}_1 \\ c \end{pmatrix}, \quad \tilde{\mathbf{v}}_2 = \begin{pmatrix} 1 \\ \tilde{\lambda}_2 \\ \frac{\rho_w + \rho}{\rho_w - \rho_s} \end{pmatrix}, \quad \tilde{\mathbf{v}}_3 = \begin{pmatrix} 1 \\ \tilde{\lambda}_3 \\ c \end{pmatrix},$$

Hence the system of partial differential equations (2.26) is strictly hyperbolic.

2.4. Numerical solution

In order to carry out the numerical solution of the stated problem we need to solve numerically the hyperbolic system of balance laws (2.23) using a well balanced numerical scheme. Moreover, the posed problem requires to solve numerically the morphological evolution modelled by equation (2.4).

2.4.1. Problem discretization

We shall consider a uniform finite volume mesh over the interval $[0, \mathcal{L}] = [x_0, x_N]$, this mesh is shown in Figure 2.3. The i -th finite volume is denoted by

$$C_i = \left(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right) = \left(x_i - \frac{x_i - x_{i-1}}{2}, x_i + \frac{x_{i+1} - x_i}{2}\right), \quad i = 1, \dots, N-1,$$

where

$$x_i = i\Delta x, \quad \text{and} \quad \Delta x = \frac{\mathcal{L}}{N},$$

the boundary finite volumes are $C_0 = (x_0, x_{\frac{1}{2}})$ and $C_N = (x_{N-\frac{1}{2}}, x_N)$.

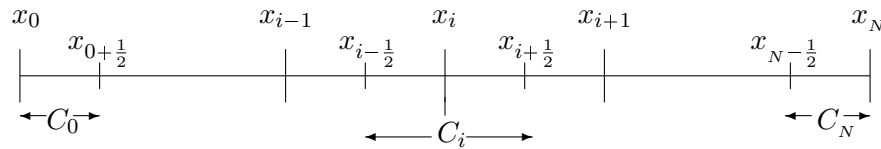


Figura 2.3: Finite volume mesh: interior finite volume C_i , boundary finite volumes C_0 and C_N .

In order to get an approximate solution $\mathbf{W}(x, t_{n+1})$ of the exact solution $\mathbf{W}(x, t)$ at time t_{n+1} , we integrate (2.23) over an arbitrary space-time rectangle $C_i \times [t_n, t_{n+1}]$:

$$\begin{aligned} \int_{C_i} \mathbf{W}(x, t_{n+1}) dx &= \int_{C_i} \mathbf{W}(x, t_n) dx - \int_{t_n}^{t_{n+1}} (\mathbf{F}(\mathbf{W}(x_{i+1/2}, t)) \\ &\quad - \mathbf{F}(\mathbf{W}(x_{i-1/2}, t))) dt + \int_{t_n}^{t_{n+1}} \int_{C_i} \mathbf{S}(x, t, \mathbf{W}(x, t)) dx dt. \end{aligned} \quad (2.30)$$

We shall denote by $\mathbf{W}_i^n$ the averaged approximate solution on the cell C_i at time t_n :

$$\mathbf{W}_i^n \cong \frac{1}{\Delta x} \int_{C_i} \mathbf{W}(x, t_n) dx, \quad (2.31)$$

then equation (2.30) takes the form

$$\begin{aligned} \mathbf{W}_i^{n+1} &= \mathbf{W}_i^n - \frac{1}{\Delta x} \int_{t_n}^{t_{n+1}} (\mathbf{F}(\mathbf{W}(x_{i+1/2}, t)) - \mathbf{F}(\mathbf{W}(x_{i-1/2}, t))) dt \\ &\quad + \int_{t_n}^{t_{n+1}} \mathbf{S}(x_i, t, \mathbf{W}_i^n) dt, \end{aligned} \quad (2.32)$$

where

$$\mathbf{S}(x_i, t, \mathbf{W}_i^n) := \frac{1}{\Delta x} \int_{C_i} \mathbf{S}(x, t, \mathbf{W}(x, t)) dx. \quad (2.33)$$

2.4.2. Extension of the Q-scheme of van Leer

In this section we shall describe briefly the extension of the Q-scheme of van Leer, put forward in [5], in order to solve (2.23).

Numerical flux

As the solution $\mathbf{W}(x, t_n)$ is defined by a piecewise function $\{\mathbf{W}_i^n\}_{i=1}^N$, the values of the approximate solution are not well defined at the boundary of each finite volume cell, so the integral of the physical flux cannot be computed at these points. For this reason, an approximation of the integral of the physical flux, given by a function Φ , also called numerical flux, is needed, so that

$$\frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \mathbf{F}(\mathbf{W}(x_{i+1/2}, t)) dt \approx \mathbf{F}_{i+\frac{1}{2}}^n := \Phi(\mathbf{W}_i^n, \mathbf{W}_{i+1}^n), \quad i = 1, \dots, N-1,$$

and

$$\frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \mathbf{F}(\mathbf{W}(x_{i-1/2}, t)) dt \approx \mathbf{F}_{i-\frac{1}{2}}^n := \Phi(\mathbf{W}_{i-1}^n, \mathbf{W}_i^n), \quad i = 1, \dots, N-1,$$

being $\Delta t = t_{n+1} - t_n$ the time step.

For the Q-scheme of van Leer the numerical flux Φ , takes the form

$$\Phi(\mathbf{U}, \mathbf{V}) = \frac{1}{2} (\mathbf{F}(\mathbf{U}) + \mathbf{F}(\mathbf{V})) - \frac{1}{2} |\mathbf{Q}(\mathbf{U}, \mathbf{V})| (\mathbf{V} - \mathbf{U}), \quad (2.34)$$

with

$$\mathbf{Q}(\mathbf{U}, \mathbf{V}) = \frac{\partial \mathbf{F}}{\partial \mathbf{W}} \left(\frac{1}{2} (\mathbf{U} + \mathbf{V}) \right).$$

The absolute value of a diagonalizable matrix $\mathbf{Q}$ is defined as follows: let $\mathbf{Q} = \mathbf{X}\mathbf{\Lambda}\mathbf{X}^{-1}$ where $\mathbf{\Lambda}$ is the diagonal matrix of the eigenvalues of $\mathbf{Q}$. Then $|\mathbf{Q}| = \mathbf{X}|\mathbf{\Lambda}|\mathbf{X}^{-1}$ where $|\mathbf{\Lambda}|$ is the diagonal matrix of the absolute value of the eigenvalues of $\mathbf{Q}$ and matrix $\mathbf{X}$ is composed by the eigenvectors of $\mathbf{Q}$.

In order to avoid the loss of viscosity in the numerical scheme, the absolute value of the eigenvalues is redefined by using the Harten regularization, [23].

Numerical source terms

In this section the way to compute the approximation of the source term is detailed.

Approximation of source terms: bed slope and sediment concentration slope

Firstly, the discretization of both the bed slope source term and the sediment concentration slope source term, $\mathbf{S}_{1,i}^n$ and $\mathbf{S}_{4,i}^n$, respectively, are described. By extending the ideas put forward in [5], these source terms are approximated as

$$\mathbf{S}_{k,i}^n := \Psi_{l,k}(x_{i-1}, x_i, t_n, \mathbf{W}_{i-1}^n, \mathbf{W}_i^n) + \Psi_{r,k}(x_i, x_{i+1}, t_n, \mathbf{W}_i^n, \mathbf{W}_{i+1}^n), \quad (2.35)$$

$$k = 1 \quad \text{or} \quad k = 4,$$

where

$$\Psi_{l,k}(x, y, t, \mathbf{U}, \mathbf{V}) = \frac{1}{2} [\mathbf{I} + |\mathbf{Q}(\mathbf{U}, \mathbf{V})| \mathbf{Q}^{-1}(\mathbf{U}, \mathbf{V})] \hat{\mathbf{S}}_k(x, y, t, \mathbf{U}, \mathbf{V}),$$

$$k = 1 \quad \text{or} \quad k = 4,$$

and

$$\Psi_{r,k}(y, z, t, \mathbf{V}, \mathbf{W}) = \frac{1}{2} [\mathbf{I} - |\mathbf{Q}(\mathbf{V}, \mathbf{W})| \mathbf{Q}^{-1}(\mathbf{V}, \mathbf{W})] \hat{\mathbf{S}}_k(y, z, t, \mathbf{V}, \mathbf{W}),$$

$$k = 1 \quad \text{or} \quad k = 4,$$

with

$$\hat{\mathbf{S}}_1(x, y, t, \mathbf{U}, \mathbf{V}) := \begin{pmatrix} 0 \\ -g \frac{h(x, t) + h(y, t)}{2} \frac{\partial z}{\partial x} \left(\frac{x + y}{2}, t \right) \\ 0 \end{pmatrix} \quad (2.36)$$

and

$$\begin{aligned} & \hat{\mathbf{S}}_4(x, y, t, \mathbf{U}, \mathbf{V}) \\ & := \begin{pmatrix} 0 \\ -\delta g \left(\frac{h(x, t) + h(y, t)}{2} \right)^2 \frac{\rho_s - \rho_w}{\rho(x, t) + \rho(y, t)} \frac{\partial c}{\partial x} \left(\frac{x + y}{2}, t \right) \\ 0 \end{pmatrix}. \end{aligned} \quad (2.37)$$

Approximation of friction slope source term To compute the friction term, it is possible to choose between an explicit time discretisation or a semi-implicit time discretisation.

- Explicit time discretization:

$$\mathbf{S}_{2,i}^n = \begin{pmatrix} 0 \\ -g \frac{\eta^2 w_{2,i}^n |w_{2,i}^n|}{(w_{1,i}^n)^{(7/3)}} \\ 0 \end{pmatrix}. \quad (2.38)$$

- Semi-implicit time discretization:

$$\mathbf{S}_{2,i}^n = \begin{pmatrix} 0 \\ -g \frac{\eta^2 w_{2,i}^{n+1} |w_{2,i}^n|}{(w_{1,i}^n)^{(7/3)}} \\ 0 \end{pmatrix}. \quad (2.39)$$

Approximation of the sediment entrainment and sediment deposition processes source terms Variables in the source term $\mathbf{S}_3(x, t, \mathbf{W}(x, t))$ are evaluated pointwise in the node i at time t_n , therefore

$$\mathbf{S}_{3,i}^n = \begin{pmatrix} \frac{E_i^n - D_i^n}{1 - p} \\ -\frac{\rho_0 - \rho}{\rho} \frac{E_i^n - D_i^n}{1 - p} \frac{w_{2,i}^n}{w_{1,i}^n} \\ E_i^n - D_i^n \end{pmatrix}, \quad (2.40)$$

where $E_i^n = E(x_i, t_n)$ and $D_i^n = D(x_i, t_n)$ whose expressions are given by (2.9) and (2.10), respectively.

Therefore the extension of the Q-scheme of van Leer to compute the approximate solution of the hyperbolic system of balance laws (2.23) takes the form

$$\mathbf{W}_i^{n+1} = \mathbf{W}_i^n - \frac{\Delta t}{\Delta x} \left(\mathbf{F}_{i+\frac{1}{2}}^n - \mathbf{F}_{i-\frac{1}{2}}^n \right) + \Delta t \sum_{k=1}^4 \mathbf{S}_{k,i}^n. \quad (2.41)$$

where the numerical flux is defined by (2.34) and the discretization for the different source terms are given by: (2.35) for bed slope and sediment concentration slope, (2.38) if the friction term is discretised in implicit way or (2.39) if the discretisation of the friction term is done in semi-implicit way. Finally, (2.40) gives the discretization for sediment entrainment and deposition process

2.4.3. HLLCS scheme

The second scheme to compute the numerical solution for the hyperbolic system of balance laws (2.23) is the HLLCS approximate Riemann solver, [33]. This scheme is based on the HLLC approximate Riemann solver [40], but the source terms $\mathbf{S}_1(x, t, \mathbf{W})$ and $\mathbf{S}_2(x, t, \mathbf{W})$ are taken into account inside the numerical flux.

Numerical flux

Following the work [33], in order to include the bed slope source term and the friction slope source term inside the numerical flux, the Rankine-Hugoniot relation for a steady shock, of shock speed $\mathcal{S}$, at $x = 0$ and $t = 0$ is considered, namely,

$$\begin{aligned} \mathbf{F}^+(\mathbf{W}(x_{i+1}, t)) - \mathbf{F}^-(\mathbf{W}(x_i, t)) - \sum_{k=1}^2 \bar{\mathbf{S}}_k \left(x_{i+\frac{1}{2}}, t, \mathbf{W}(x_{i+\frac{1}{2}}, t) \right) \\ = \mathcal{S} (\mathbf{W}^+(x_{i+1}, t) - \mathbf{W}^-(x_i, t)) = 0, \end{aligned} \quad (2.42)$$

where $\bar{\mathbf{S}}_k$ ($k = 1, 2$), is the averaged approximated source term at $x = 0$ and $t = 0$, so that

$$\sum_{k=1}^2 \bar{\mathbf{S}}_k(x_{i+\frac{1}{2}}, t, \mathbf{W}(x_{i+\frac{1}{2}}, t)) = \begin{bmatrix} \bar{\mathcal{S}}_1 \\ \bar{\mathcal{S}}_2 \\ \bar{\mathcal{S}}_3 \end{bmatrix}. \quad (2.43)$$

As the source terms $\bar{\mathbf{S}}_1(x, t, \mathbf{W})$ and $\bar{\mathbf{S}}_2(x, t, \mathbf{W})$ are taken into account inside the numerical flux, the hyperbolic system of balance laws (2.23) can be solved by using the numerical scheme

$$\begin{aligned} \mathbf{W}(x_i, t_{n+1}) = \mathbf{W}(x_i, t_n) - \frac{\Delta t}{\Delta x} \left(\mathbf{F}_{i+\frac{1}{2}}^{HLLCS-} - \mathbf{F}_{i-\frac{1}{2}}^{HLLCS+} \right) \\ + \Delta t \sum_{k=3}^4 \mathbf{S}_k(x_i, t_n, \mathbf{W}(x_i, t_n)), \end{aligned} \quad (2.44)$$

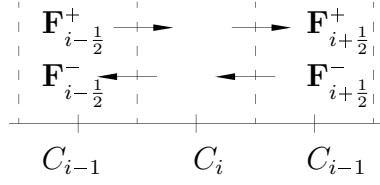


Figura 2.4: Numerical flux in the cell C_i for HLLCS scheme.

where $\mathbf{F}_{i\pm\frac{1}{2}}^{HLLCS\pm}$ are the numerical flows with the source terms inside it. The contributions made by the numerical flux in the cells depend on the sign of the eigenvalues at the boundaries, see Figure 2.4.

In order to define the numerical flux we need to have established values λ_i , λ_{i+1} , and $\lambda_*^\pm$, which represent a middle wave related to the source terms $\bar{\mathbf{S}}_1$ and $\bar{\mathbf{S}}_2$, see Figure 2.5. These values are given by expressions (2.49)-(2.54).

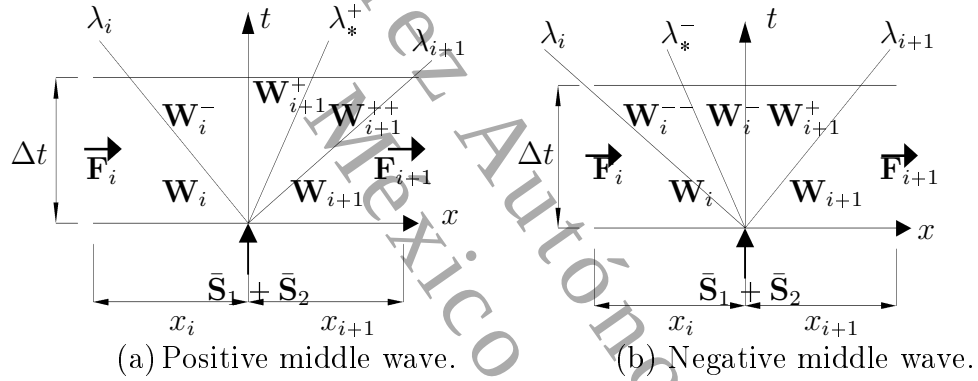


Figura 2.5: HLLCS numerical flux.

In order to compute the numerical flux, we shall extend for the bottom variations the work done in [33], where the numerical flux is computed as:

- If $0 \leq \lambda_i$, then

$$\begin{aligned} \mathbf{F}_{i+\frac{1}{2}}^{HLLCS-} &= \mathbf{F}(\mathbf{W}(x_i, t)), \\ \mathbf{F}_{i+\frac{1}{2}}^{HLLCS+} &= \mathbf{F}(\mathbf{W}(x_i, t)) + \sum_{k=1}^2 \bar{\mathbf{S}}_k(x_{i+\frac{1}{2}}, t, \mathbf{W}(x_{i+\frac{1}{2}}, t)). \end{aligned} \quad (2.45)$$

- If $0 \geq \lambda_{i+1}$, then

$$\begin{aligned} \mathbf{F}_{i+\frac{1}{2}}^{HLLCS-} &= \mathbf{F}(\mathbf{W}(x_{i+1}, t)) - \sum_{k=1}^2 \bar{\mathbf{S}}_k(x_{i+\frac{1}{2}}, t, \mathbf{W}(x_{i+\frac{1}{2}}, t)), \\ \mathbf{F}_{i+\frac{1}{2}}^{HLLCS+} &= \mathbf{F}(\mathbf{W}(x_{i+1}, t)). \end{aligned} \quad (2.46)$$

- If $\lambda_i \leq 0 \leq \lambda_{i+1}$, then

- if $\lambda_*^+ > 0$, then

$$\begin{aligned} \mathbf{F}_{i+\frac{1}{2}}^{HLLCS-} &= \mathbf{F}(\mathbf{W}(x_i, t)) + \lambda_i(\mathbf{W}^-(x_i, t) - \mathbf{W}(x_i, t)), \\ \mathbf{F}_{i+\frac{1}{2}}^{HLLCS+} &= \mathbf{F}_{i+\frac{1}{2}}^{HLLCS-} + \sum_{k=1}^2 \bar{\mathbf{S}}_k(x_{i+\frac{1}{2}}, t, \mathbf{W}(x_{i+\frac{1}{2}}, t)), \end{aligned} \quad (2.47)$$

where

$$\begin{aligned} \mathbf{W}^-(x_i, t) &= \mathbf{W}^+(x_{i+1}, t) - \bar{\mathbf{H}}_{i+\frac{1}{2}}^+, \\ \bar{\mathbf{H}}_{i+\frac{1}{2}}^+ &= -\frac{\bar{\mathbf{S}}_2}{\tilde{\lambda}_1 \tilde{\lambda}_2} \begin{pmatrix} 1 \\ 0 \\ c(x_i, t) \end{pmatrix}, \quad \mathbf{W}^+(x_{i+1}, t) = w_{1,i+1}^+ \begin{pmatrix} 1 \\ \lambda_*^+ \\ c(x_i, t) \end{pmatrix}, \end{aligned}$$

and

$$w_{1,i+1}^+ = \left(\frac{(w_{1,i})(u_i - \lambda_i) - \lambda_i \bar{H}_{i+\frac{1}{2},1}^+}{\lambda_*^+ - \lambda_i} \right).$$

- If $\lambda_*^- < 0$, then

$$\begin{aligned} \mathbf{F}_{i+1/2}^{HLLCS-} &= \mathbf{F}(\mathbf{W}(x_{i+1}, t)) + \lambda_{i+1}(\mathbf{W}^+(x_{i+1}, t) - \mathbf{W}(x_{i+1}, t)), \\ \mathbf{F}_{i+1/2}^{HLLCS+} &= \mathbf{F}_{i+1/2}^{HLLCS-} - \sum_{k=1}^2 \bar{\mathbf{S}}_k(x_{i+\frac{1}{2}}, t, \mathbf{W}(x_{i+\frac{1}{2}}, t)), \end{aligned} \quad (2.48)$$

where

$$\begin{aligned} \mathbf{W}^+(x_{i+1}, t) &= \mathbf{W}^-(x_i, t) + \bar{\mathbf{H}}_{i+\frac{1}{2}}^-, \\ \bar{\mathbf{H}}_{i+\frac{1}{2}}^- &= -\frac{\bar{\mathbf{S}}_2}{\tilde{\lambda}_1 \tilde{\lambda}_2} \begin{pmatrix} 1 \\ 0 \\ c(x_{i+1}, t) \end{pmatrix}, \quad \mathbf{W}^-(x_i, t) = w_{1,i}^- \begin{pmatrix} 1 \\ \lambda_*^- \\ c(x_{i+1}, t) \end{pmatrix}, \end{aligned}$$

and

$$w_{1,i}^- = \left(\frac{w_{1,i+1}(u_{i+1} - \lambda_{i+1}) - \lambda_{i+1} \bar{H}_{i+\frac{1}{2},1}^-}{\lambda_*^- - \lambda_{i+1}} \right).$$

The values λ_i, λ_{i+1} are given by the following expressions:

$$\lambda_i = \begin{cases} \min \left\{ \tilde{\lambda}_1; u_i - \sqrt{gw_{1,i}}; u_{i+1} - \sqrt{gw_{1,i+1}} \right\} & \text{if } |\mathcal{S}_2| = 0, \\ \tilde{\lambda}_1 & \text{if } |\mathcal{S}_2| \neq 0. \end{cases} \quad (2.49)$$

$$\lambda_{i+1} = \begin{cases} \max \left\{ \tilde{\lambda}_2; u_i + \sqrt{gw_{1,i}}; u_{i+1} + \sqrt{gw_{1,i+1}} \right\} & \text{if } |\mathcal{S}_2| = 0, \\ \tilde{\lambda}_2 & \text{if } |\mathcal{S}_2| \neq 0. \end{cases} \quad (2.50)$$

The average values $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ are the Roe mean [35], with expressions (2.51) and (2.52).

$$\tilde{u} = \frac{u_i \sqrt{w_{1,i}} + u_{i+1} \sqrt{w_{1,i+1}}}{\sqrt{w_{1,i}} + \sqrt{w_{1,i+1}}} \quad (2.51)$$

and

$$\tilde{\lambda}_1 = \tilde{u} - \sqrt{g \frac{w_{1,i} + w_{1,i+1}}{2}}, \quad \tilde{\lambda}_2 = \tilde{u} + \sqrt{g \frac{w_{1,i} + w_{1,i+1}}{2}}. \quad (2.52)$$

Finally the values for $\lambda_*^\pm$ take the form

$$\lambda_*^+ = \frac{\lambda_i w_{1,i+1} (u_{i+1} - \lambda_{i+1}) - \lambda_{i+1} w_{1,i} (u_i - \lambda_i) + \lambda_{i+1} (\lambda_i \bar{H}_{i+\frac{1}{2},1}^+ - \mathcal{S}_1)}{w_{1,i+1} (u_{i+1} - \lambda_{i+1}) - w_{1,i} (u_i - \lambda_i) + \lambda_i \bar{H}_{i+\frac{1}{2},1}^+}, \quad (2.53)$$

and

$$\lambda_*^- = \frac{\lambda_i w_{1,i+1} (u_{i+1} - \lambda_{i+1}) - \lambda_{i+1} w_{1,i} (u_i - \lambda_i) + \lambda_{i+1} (\lambda_i \bar{H}_{i+\frac{1}{2},1}^- - \mathcal{S}_1)}{w_{1,i+1} (u_{i+1} - \lambda_{i+1}) - w_{1,i} (u_i - \lambda_i) + \lambda_i \bar{H}_{i+\frac{1}{2},1}^-}. \quad (2.54)$$

Where $\bar{\mathcal{S}}_k$ are the components of the source terms evaluated on the Roe mean at the boundary $x_{i+\frac{1}{2}}$, which are given by (2.51) and (2.52), and $\mathcal{S}_k$ is the pointwise evaluation of these components in the cell C_i .

Bed slope approximation

In order to fully describe the discretization of bed slope source term, we shall follow [33], where the approximation for the bed slope is given by

$$\frac{\partial z}{\partial x}(x_{i+\frac{1}{2}}, t_n) \approx \frac{g}{\Delta x} \left(w_{1,j}^n - \frac{|\delta z'|}{2} \right) \delta z', \quad (2.55)$$

where

$$j = \begin{cases} i & \text{if } \delta z \geq 0, \\ i+1 & \text{if } \delta z < 0, \end{cases} \quad \text{and} \quad \delta z' = \begin{cases} w_{1,i}^n & \text{if } \delta z \geq 0 \text{ and } d_i < z_{i+1}^n, \\ w_{1,i+1}^n & \text{if } \delta z < 0 \text{ and } d_{i+1} < z_i^n, \\ \delta z & \text{else,} \end{cases}$$

being $\delta z = z(x_{i+1}, t_n) - z(x_i, t_n)$ and $d_i = w_1(x_i, t_n) + z(x_i, t_n)$.

Friction term approximation

The friction term is approximated by using the following relation, (see [33]).

$$\left| \frac{S_f}{g\tilde{w}_1} \Delta x \right|_{i+\frac{1}{2}} = \min \left(\left| \frac{S_f}{g\tilde{w}_1} \Delta x \right|, \left| \tilde{u} \frac{|u_{\min}|}{2g} \right| \right)_{i+\frac{1}{2}}, \quad (2.56)$$

where

$$|u_{\min}| = \min(|u_i|, |u_{i+1}|), \quad (2.57)$$

and S_f is computed using equation (2.8) by choosing c_f as the friction coefficient.

Approximation of sediment concentration slope, sediment erosion and sediment deposition process

The source terms $\mathbf{S}_3(x, t)$ and $\mathbf{S}_4(x, t)$ in the numerical scheme (2.44) are approximated as follows

$$\sum_{k=3}^4 \mathbf{S}_{k,i}^n = \begin{pmatrix} \frac{E_i^n - D_i^n}{1-p} \\ -\delta g w_{1,i}^2 \frac{\rho_s - \rho_w c_i^n - c_{i-1}^n}{2\rho_i} \frac{\Delta x}{\Delta x} - \frac{\rho_0 - \rho}{\rho} \frac{E_i^n - D_i^n}{1-p} u_i^n \\ E_i^n - D_i^n \end{pmatrix}. \quad (2.58)$$

2.4.4. Numerical scheme for the non-conservative system

In order to deal with the presences of non conservative products, a suitable numerical treatment of them is needed. Following [10] the numerical solution of (2.26) at cell C_i at time t_n is computed using a numerical scheme of the form

$$\begin{aligned} \mathbf{W}_i^{n+1} = & \mathbf{W}_i^n - \frac{\Delta t}{\Delta x} \left(\mathcal{A}_D^+ \left(\mathbf{W}_{i-\frac{1}{2}}^n \right) (\mathbf{W}_i^n - \mathbf{W}_{i-1}^n) \right. \\ & \left. + \mathcal{A}_D^- \left(\mathbf{W}_{i+\frac{1}{2}}^n \right) (\mathbf{W}_{i+1}^n - \mathbf{W}_i^n) \right) + \Delta t \sum_{k=1}^3 \mathbf{S}_{k,i}^n, \end{aligned} \quad (2.59)$$

where

$$\mathcal{A}_D^\pm \left(\mathbf{W}_{i+\frac{1}{2}}^n \right) = \frac{1}{2} \left(\mathcal{A} \left(\mathbf{W}_{i+\frac{1}{2}}^n \right) \pm \left| \mathcal{A} \left(\mathbf{W}_{i+\frac{1}{2}}^n \right) \right| \right),$$

Taking into account the definition of $\mathcal{A}$ given by equation (2.29) the numerical scheme (2.59) becomes

$$\begin{aligned} \mathbf{W}_i^{n+1} = & \mathbf{W}_i^n - \frac{\Delta t}{\Delta x} \left(\mathcal{G}_{i+\frac{1}{2}}^n - \mathcal{G}_{i-\frac{1}{2}}^n \right) \\ & - \frac{\Delta t}{2\Delta x} \left(\mathbf{B} \left(\mathbf{W}_{i+\frac{1}{2}}^n \right) (\mathbf{W}_{i+1}^n - \mathbf{W}_i^n) + \mathbf{B} \left(\mathbf{W}_{i-\frac{1}{2}}^n \right) (\mathbf{W}_i^n - \mathbf{W}_{i-1}^n) \right) \\ & + \Delta t \sum_{k=1}^3 \mathbf{S}_{k,i}^n \end{aligned} \quad (2.60)$$

where

$$\mathcal{G}_{i+\frac{1}{2}}^n = \frac{1}{2} (\mathbf{F}(\mathbf{W}_{i+1}^n) + \mathbf{F}(\mathbf{W}_i^n)) - \frac{1}{2} \left| \mathcal{A}(\mathbf{W}_{i+\frac{1}{2}}^n) \right| (\mathbf{W}_{i+1}^n - \mathbf{W}_i^n). \quad (2.61)$$

The discretization of the bed slope source term $\mathbf{S}_1(x, t)$ is carried out following the methodology explained in section 2.4.2, but in this case the upwind process is done with regard to the eigenvalues of matrix $\mathcal{A}$.

The source term, which contains the friction slope $\mathbf{S}_2(x, t)$, is discretized by using either, (2.38) or (2.39) while the source term that takes into account the sediment entrainment and deposition processes $\mathbf{S}_3(x, t)$, is discretized by using (2.40).

2.4.5. Morphological evolution

The Exner equation (2.4) is not actually hyperbolic, however it is possible to write a wave speed estimation $\tilde{\lambda}_b$ associated to the sediment flux (see [25]), such that the Exner equation becomes

$$\frac{\partial z}{\partial t}(x, t) + \tilde{\lambda}_b \frac{\partial z}{\partial x}(x, t) = \frac{D(x, t) - E(x, t)}{1 - p}, \quad (2.62)$$

where

$$\tilde{\lambda}_b = \frac{1}{1 - p} \frac{\partial q_b}{\partial z}. \quad (2.63)$$

This wave speed is the basis to build an upwind strategy to solve the morphological evolution.

In order to solve the Exner equation (2.4), we integrate that equation over a space-time arbitrary rectangle $C_i \times [t_n, t_{n+1}]$, as it was done in section 2.4.1.

$$z_i^{n+1} = z_i^n + \frac{\Delta t}{1 - p} (D_i^n - E_i^n) - \frac{1}{\Delta x} \int_{t_n}^{t_{n+1}} \frac{1}{1 - p} \left(q_b(x_{i+\frac{1}{2}}, t) - q_b(x_{i-\frac{1}{2}}, t) \right) dt. \quad (2.64)$$

Following the finite volume methodology, an approximation $q_{b,i\pm\frac{1}{2}}^n$ for the integral of the bedload discharge at the common boundary $x_{i\pm\frac{1}{2}}$ between two finite volume cells, is needed, namely

$$q_{b,i\pm\frac{1}{2}}^n \approx \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} q_b(x_{i\pm\frac{1}{2}}, t) dt. \quad (2.65)$$

Following [25], this approximation is defined as follows

$$\begin{aligned} q_{b,i+\frac{1}{2}}^n &:= \begin{cases} q_b(x_i, t_n) & \text{if } \tilde{\lambda}_{b,i+\frac{1}{2}} > 0, \\ q_b(x_{i+1}, t_n) & \text{if } \tilde{\lambda}_{b,i+\frac{1}{2}} < 0, \end{cases} \\ q_{b,i-\frac{1}{2}}^n &:= \begin{cases} q_b(x_{i-1}, t_n) & \text{if } \tilde{\lambda}_{b,i-\frac{1}{2}} > 0, \\ q_b(x_i, t_n) & \text{if } \tilde{\lambda}_{b,i-\frac{1}{2}} < 0, \end{cases} \end{aligned} \quad (2.66)$$

where

$$\tilde{\lambda}_{b,i+\frac{1}{2}} = \frac{1}{1-p} \frac{q_b(x_{i+1}, t_n) - q_b(x_i, t_n)}{z(x_{i+1}, t_n) - z(x_i, t_n)}. \quad (2.67)$$

Finally, the scheme to solve the Exner equation (2.4) reads

$$z_i^{n+1} = z_i^n + \frac{\Delta t}{1-p} \left(D_i^n - E_i^n - \frac{1}{\Delta x} \left(q_{b,i+\frac{1}{2}}^n - q_{b,i-\frac{1}{2}}^n \right) \right). \quad (2.68)$$

2.4.6. Numerical treatment of boundary conditions

The numerical methods described earlier are applied to internal cells, C_i , $i = 1, \dots, N-1$. To compute the values of the conservative variables at the boundaries, fictitious neighbouring nodes $\hat{x}_B$ (see [4]) or ghost cells C_B , (see[26]), are introduced, see Figure 2.6.

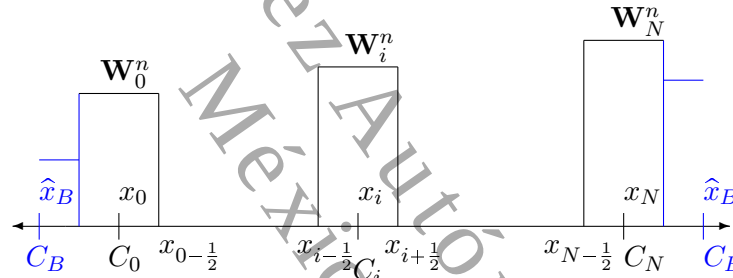


Figura 2.6: Fictitious neighbouring nodes, $\hat{x}_B$, and ghost cells, C_B .

The values of the conservative variables in these cells are defined according to the boundary conditions: inflow or outflow, the values of the conservative variables at ghost cells are equal to the values at the boundary cells. Wall, the value of the velocity at the ghost cells is the opposite of velocity at the boundary cells, the remains values at ghost cell are equal to the values at the boundary cell.

The approximate solution at the boundaries is computed by using the values of the conservative variables on ghost cells and either of the numerical schemes described earlier.

2.4.7. Steady stationary solution

A numerical scheme is called well balanced if it can preserve a family of stationary solution. In order to get the a dynamic equilibrium state for the system of balance laws (2.23) and (2.4), let us suppose that $u(x, t) = 0$, in this case $\theta < \theta_c$, then the system of balance laws reduces to

$$\frac{\partial h}{\partial t}(x, t) = \frac{-1}{1-p} D(x, t), \quad (2.69)$$

$$\frac{\partial h}{\partial x}(x, t) + \frac{\partial z}{\partial x}(x, t) = 0, \quad (2.70)$$

$$\frac{\partial hc}{\partial t}(x, t) = -D(x, t), \quad (2.71)$$

$$\frac{\partial z}{\partial t}(x, t) = \frac{1}{1-p} D(x, t). \quad (2.72)$$

From equations (2.69) and (2.72) we can see that both, the depth of water and the depth of the bottom may vary, but the free surface of water $\varsigma(x, t) = h(x, t) + z(x, t)$ stays constant with respect of time, namely

$$\frac{\partial \varsigma}{\partial t}(x, t) = \frac{\partial h}{\partial t}(x, t) + \frac{\partial z}{\partial t}(x, t) = 0. \quad (2.73)$$

Therefore, from (2.70) and (2.73) is concluded that the free surface of water is a constant with respect of time and space.

From equation (2.71) and by using (2.69), we get

$$\frac{\partial c}{\partial t}(x, t) = \frac{D(x, t)}{h(x, t)} \left(\frac{c(x, t)}{1-p} - 1 \right). \quad (2.74)$$

This equation shows that the variation of the volumetric sediment concentration over time is equal to zero if either $D(x, t) = 0$ then $c(x, t) = 0$ or $c(x, t) = 1 - p$. Moreover if $c(x, t) < 1 - p$ then $c(x, t)$ is a decreasing function over time.

Definition 1 *The dynamic equilibrium state of balance laws system (2.23) and (2.4) is given by the following conditions*

- *The water free surface is a constant function, see equations (2.70) and (2.73).*
- *Null velocity in the whole domain, $u = 0$.*
- *The variation of suspended sediment concentration over time is equal to the opposite of deposition, see equation (2.71).*

Proposition 1 *Let us suppose that $\mathbf{W}(x_i, t_n)$ and $z(x_i, t_n)$ fulfill the dynamic equilibrium state conditions, then the numerical scheme defined by equations (2.41), (2.34), (2.35), either (2.38) or (2.39), (2.40) and (2.68), computes the dynamic equilibrium state conditions exactly, that is*

- *The free surface is constant.*
- *The velocity is null over whole domain.*
- *The suspended sediment concentration variation with respect to time is equal to opposite of deposition.*

Demostración.

The values for the different variables at interface $x_{i+1/2}$ will be computed as the arithmetic mean. Under water at rest hypothesis, the different elements of the numerical scheme take the following forms:

- Convective discretization,

$$\begin{aligned} & \frac{1}{\Delta x} (\mathbf{F}_{i+\frac{1}{2}}^n - \mathbf{F}_{i-\frac{1}{2}}^n) \\ &= \begin{pmatrix} \frac{w_{1,i}^n - w_{1,i-1}^n}{2\Delta x} \sqrt{g w_{1,i-\frac{1}{2}}^n} - \frac{w_{1,i+1}^n - w_{1,i}^n}{2\Delta x} \sqrt{g w_{1,i+\frac{1}{2}}^n} \\ \frac{g}{4\Delta x} ((w_{1,i+1}^n)^2 - (w_{1,i-1}^n)^2) \\ c_{i-\frac{1}{2}}^n \frac{w_{1,i}^n - w_{1,i-1}^n}{2\Delta x} \sqrt{g w_{1,i-\frac{1}{2}}^n} - c_{i+\frac{1}{2}}^n \frac{w_{1,i+1}^n - w_{1,i}^n}{2\Delta x} \sqrt{g w_{1,i+\frac{1}{2}}^n} \end{pmatrix}. \end{aligned} \quad (2.75)$$

- Discretization of bed slope source term,

$$\mathbf{S}_{1,i}^n = \begin{pmatrix} \frac{1}{2} \frac{z_{i+1}^n - z_i^n}{\Delta x} \sqrt{g w_{1,i+\frac{1}{2}}^n} - \frac{1}{2} \frac{z_i^n - z_{i-1}^n}{\Delta x} \sqrt{g w_{1,i-\frac{1}{2}}^n} \\ -\frac{g}{2} w_{1,i+\frac{1}{2}}^n \frac{z_{i+1}^n - z_i^n}{\Delta x} - \frac{g}{2} w_{1,i-\frac{1}{2}}^n \frac{z_i^n - z_{i-1}^n}{\Delta x} \\ \frac{1}{2} c_{i+\frac{1}{2}}^n \sqrt{g w_{1,i+\frac{1}{2}}^n} \frac{z_{i+1}^n - z_i^n}{\Delta x} - \frac{1}{2} c_{i-\frac{1}{2}}^n \sqrt{g w_{1,i-\frac{1}{2}}^n} \frac{z_i^n - z_{i-1}^n}{\Delta x} \end{pmatrix}. \quad (2.76)$$

- Discretization of friction slope source term, $\mathbf{S}_{2,i}^n = \mathbf{0}$.
- Discretization of sediment entrainment and sediment deposition process source term,

$$\mathbf{S}_{3,i}^n = \begin{pmatrix} \frac{-D_i^n}{1-p} \\ 0 \\ -D_i^n \end{pmatrix}. \quad (2.77)$$

- Discretization of sediment concentration slope source term,

$$\mathbf{S}_{4,i}^n = \delta \begin{pmatrix} S_{(4,i),1}^n \\ S_{(4,i),2}^n \\ S_{(4,i),3}^n \end{pmatrix}, \quad (2.78)$$

where

$$\begin{aligned}
 S_{(4,i),1}^n &= \frac{1}{2} \frac{c_{i+1}^n - c_i^n}{\Delta x} \sqrt{g w_{1,i+\frac{1}{2}}^n} w_{1,i+\frac{1}{2}}^n \frac{\rho_s - \rho_w}{2\rho_{i+\frac{1}{2}}^n} \\
 &\quad - \frac{1}{2} \frac{c_i^n - c_{i-1}^n}{\Delta x} \sqrt{g w_{1,i-\frac{1}{2}}^n} w_{1,i-\frac{1}{2}}^n \frac{\rho_s - \rho_w}{2\rho_{i-\frac{1}{2}}^n}, \\
 S_{(4,i),2}^n &= -\frac{g}{2} \frac{c_{i+1}^n - c_i^n}{\Delta x} \left(w_{1,i+\frac{1}{2}}^n\right)^2 \frac{\rho_s - \rho_w}{2\rho_{i+\frac{1}{2}}^n} \\
 &\quad - \frac{g}{2} \frac{c_i^n - c_{i-1}^n}{\Delta x} \left(w_{1,i-\frac{1}{2}}^n\right)^2 \frac{\rho_s - \rho_w}{2\rho_{i-\frac{1}{2}}^n}, \\
 S_{(4,i),3}^n &= \frac{1}{2} \frac{c_{i+1}^n - c_i^n}{\Delta x} c_{1+\frac{1}{2}}^n w_{1,i+\frac{1}{2}}^n \sqrt{g w_{1,i+\frac{1}{2}}^n} \frac{\rho_s - \rho_w}{2\rho_{i+\frac{1}{2}}^n} \\
 &\quad - \frac{1}{2} \frac{c_i^n - c_{i-1}^n}{\Delta x} c_{i-\frac{1}{2}}^n w_{1,i-\frac{1}{2}}^n \sqrt{g w_{1,i-\frac{1}{2}}^n} \frac{\rho_s - \rho_w}{2\rho_{i-\frac{1}{2}}^n}.
 \end{aligned}$$

- Exner equation discretization,

$$z_i^{n+1} = z_i^n + \Delta t \frac{D_i^n}{1-p}$$

Let ς_i^{n+1} be the approximation of the free surface of water computed by the numerical scheme (2.41), therefore

$$\begin{aligned}
 \varsigma_i^{n+1} &= w_{1,i}^{n+1} + z_i^{n+1} \\
 &= w_{1,i}^n - \frac{\Delta t}{2} \left(\frac{w_{1,i}^n - w_{1,i-1}^n}{\Delta x} \sqrt{g w_{1,i-\frac{1}{2}}^n} - \frac{w_{1,i+1}^n - w_{1,i}^n}{\Delta x} \sqrt{g w_{1,i+\frac{1}{2}}^n} \right) \\
 &\quad + \frac{\Delta t}{2} \left(\frac{z_{i+1}^n - z_i^n}{\Delta x} \sqrt{g w_{1,i+\frac{1}{2}}^n} - \frac{z_i^n - z_{i-1}^n}{\Delta x} \sqrt{g w_{1,i-\frac{1}{2}}^n} \right) \\
 &\quad + \delta \frac{\Delta t}{2} \left(\frac{c_{i+1}^n - c_i^n}{\Delta x} w_{1,i+\frac{1}{2}}^n \sqrt{g w_{1,i+\frac{1}{2}}^n} \frac{\rho_s - \rho_w}{2\rho_{i+\frac{1}{2}}^n} \right. \\
 &\quad \left. - \frac{c_i^n - c_{i-1}^n}{\Delta x} w_{1,i-\frac{1}{2}}^n \sqrt{g w_{1,i-\frac{1}{2}}^n} \frac{\rho_s - \rho_w}{2\rho_{i-\frac{1}{2}}^n} \right) - \Delta t \frac{D_i^n}{1-p} + z_i^n + \Delta t \frac{D_i^n}{1-p}.
 \end{aligned} \tag{2.79}$$

From (2.79) and taking into account that for this case $\delta = 0$, we get

$$\varsigma_i^{n+1} = \varsigma_i^n - \frac{\Delta t}{2} \left[\sqrt{g w_{1,i-\frac{1}{2}}^n} \left(\frac{\varsigma_i^n - \varsigma_{i-1}^n}{\Delta x} \right) - \sqrt{g w_{1,i+\frac{1}{2}}^n} \left(\frac{\varsigma_{i+1}^n - \varsigma_i^n}{\Delta x} \right) \right]. \tag{2.80}$$

Bearing in mind that at time $t = t_n$ the free surface is a constant function with respect of x , then we can conclude that $\varsigma_i^{n+1} = \varsigma_i^n$, that is to say, the free surface is a constant function.

The computed unit discharge by the numerical scheme (2.41) at time t_{n+1} , reads

$$\begin{aligned} w_{2,i}^{n+1} = & w_{2,i}^n - \frac{g\Delta t}{4\Delta x} ((w_{1,i+1}^n)^2 - (w_{1,i-1}^n)^2) - \frac{g\Delta t}{2} \left(w_{1,i+\frac{1}{2}} \frac{z_{i+1}^n - z_i^n}{\Delta x} \right. \\ & + w_{1,i-\frac{1}{2}} \frac{z_i^n - z_{i-1}^n}{\Delta x} \Big) - \delta \frac{g\Delta t}{2} \left(\frac{c_{i+1}^n - c_i^n}{\Delta x} w_{1,i+\frac{1}{2}}^2 \frac{\rho_s - \rho_w}{2\rho_{i+\frac{1}{2}}^n} \right. \\ & \left. + \frac{c_i^n - c_{i-1}^n}{\Delta x} w_{1,i-\frac{1}{2}}^2 \frac{\rho_s - \rho_w}{2\rho_{i-\frac{1}{2}}^n} \right). \end{aligned} \quad (2.81)$$

Taking into account that $w_{1,i\pm\frac{1}{2}}$ is the arithmetic mean between the states, as well, considering that $\delta = 0$, $z_i^n = \varsigma_i^n - w_{1,i}^n$ and $\varsigma_{n+1}^n = \varsigma_n^n$, the equation (2.81) reads

$$\begin{aligned} w_{2,i}^{n+1} = & w_{2,i}^n - \frac{g\Delta t}{4\Delta x} ((w_{1,i+1}^n)^2 - (w_{1,i-1}^n)^2) \\ & - \frac{g\Delta t}{2} \left(\frac{w_{1,i}^n + w_{1,i+1}^n}{2} \frac{w_{1,i}^n - w_{1,i+1}^n}{\Delta x} + \frac{w_{1,i}^n + w_{1,i-1}^n}{2} \frac{w_{1,i-1}^n - w_{1,i}^n}{\Delta x} \right). \end{aligned} \quad (2.82)$$

Straightforward computations allow us conclude that $w_{2,i}^{n+1} = w_{2,i}^n$, which means that the numerical scheme compute null velocity on whole spatial domain.

The suspended sediment concentration computed by the numerical scheme (2.41) takes the form

$$\begin{aligned} w_{3,i}^{n+1} = & w_{3,i}^n - \frac{\Delta t}{2} \left(c_{i-\frac{1}{2}}^n \frac{w_{1,i}^n - w_{1,i-1}^n}{\Delta x} \sqrt{g w_{1,i-\frac{1}{2}}^n} - c_{i+\frac{1}{2}}^n \frac{w_{1,i+1}^n - w_{1,i}^n}{\Delta x} \sqrt{g w_{1,i+\frac{1}{2}}^n} \right) \\ & + \frac{\Delta t}{2} \left(c_{i+\frac{1}{2}}^n \sqrt{g w_{1,i+\frac{1}{2}}^n} \frac{z_{i+1}^n - z_i^n}{\Delta x} - c_{i-\frac{1}{2}}^n \sqrt{g w_{1,i-\frac{1}{2}}^n} \frac{z_i^n - z_{i-1}^n}{\Delta x} \right) \\ & + \delta \Delta t \left(\frac{c_{i+1}^n - c_i^n}{\Delta x} c_{i+\frac{1}{2}}^n w_{1,i+\frac{1}{2}}^n \sqrt{g w_{1,i+\frac{1}{2}}^n} \frac{\rho_s - \rho_w}{2\rho_{i+\frac{1}{2}}^n} - \frac{c_i^n - c_{i-1}^n}{\Delta x} \right. \\ & \left. c_{i-\frac{1}{2}}^n w_{1,i-\frac{1}{2}}^n \sqrt{g w_{1,i-\frac{1}{2}}^n} \frac{\rho_s - \rho_w}{2\rho_{i-\frac{1}{2}}^n} \right) - \Delta t D_i^n. \end{aligned} \quad (2.83)$$

Considering that under water at rest supposition $\delta = 0$, $z_i^n = \varsigma_i^n - w_{1,i}^n$ and $\varsigma_{n+1}^n = \varsigma_n^n$, equation (2.83) becomes in

$$\frac{w_{3,i}^{n+1} - w_{3,i}^n}{\Delta t} = -D_i^n. \quad (2.84)$$

From equations (2.80), (2.82) and (2.84) we can conclude the proof of proposition 1. ■ A similar proof can be done for the numerical scheme (2.60) which implements the non conservative form given by equation (2.28).

2.4.8. Wet-dry fronts

In shallow water flow simulations (e.g. floods, dam break, unsteady river flow) wet-dry fronts appear, where a wet region of the spatial domain becomes dry or vice versa. In these situations a boundary movement, defined by the wet-dry front, is produced. In order to obtain accurate, consistent results a suitable treatment is needed, [6].

On wet-dry fronts one or two conditions are imposed (see [12]). The first is the bed slope redefinition, which is made to obtain the exact balance between the wet-dry front and the pressure terms for the hydrostatic flux. If the bed slope is not redefined then spurious waves can appear. The second condition is of reflection.

In order to impose these conditions, a wet-dry parameter ε_{wd} is defined. Let us suppose that C_i and C_j are the wet cell and dry cell, respectively.

The balance condition (see [6]), is verified if

$$(\Delta z)_{i,j} = - \left((\Delta w_1)_{i,j} + (w_1)_{i,j} \left(\frac{\rho_s - \rho_w}{2\rho} \right) (\Delta c)_{i,j} \right). \quad (2.85)$$

On wet-dry fronts this condition is not verified, see Figure 2.7, and therefore the bed slope redefinition is needed, in order to do that, the rise of the bed is redefined as follows.

- Q-scheme of van Leer (2.41), Numerical scheme for non-conservative form (2.60).

$$\Delta z := \begin{cases} w_{1,i} - w_{1,j} + \frac{w_{1,i} + w_{1,j}}{2} \frac{\rho_s - \rho_w}{2\rho_{i,j}} (c_j - c_i) & \text{if } w_{1,j} < \varepsilon_{wd} \text{ and} \\ & w_{1,i} - w_{1,j} < z_j - z_i, \\ z_j - z_i & \text{else.} \end{cases} \quad (2.86)$$

Generally, the velocity at neighboring cells of the wet-dry front is different from zero. In this situation the reflection condition imposes a unit averaged-flux discharge equal to zero at the wet-dry boundary.

- HLLCS Riemann solver. Since the accurate behaviour is obtained a similar way as the bed slope was approximated, (2.55), a redefinition of rise of the bed is not needed.

Regarding the morphological changes, since the water depth appears in the denominator of the sediment entrainment formula, low water depth values might cause great erosion without physical sense, which can cause numerical instabilities, ([29]). The way to deal with this problem is the following:

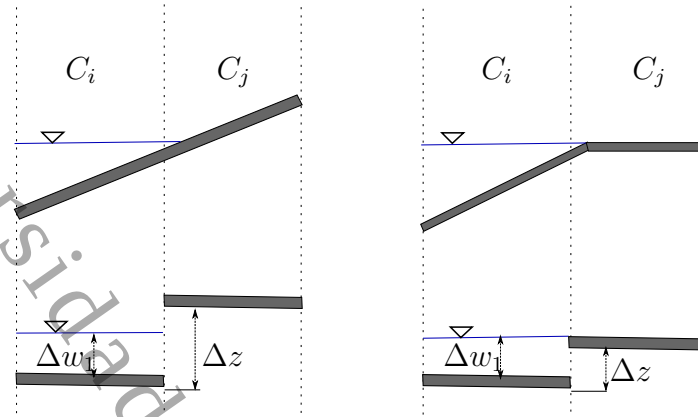


Figura 2.7: Treatment of wet-dry fronts when both the scheme which implements the Q-scheme of van Leer and the non-conservative system are used.

- Q-scheme of van Leer and scheme of non-conservative form. A threshold value ε_{ED} is defined, so that, if the water depth in the cell C_j is less than ε_{ED} , then the sediment entrainment is fixed at zero, [29].
- HLLCS Riemann solver. An interval $(\mathcal{H}_0, \mathcal{H}_1)$ is established, to ensure that the volumetric sediment concentration always is analyzed in wet regions. If the water depth ranges from $\mathcal{H}_0$ to $\mathcal{H}_1$, then the values for sediment entrainment $E(x, t)$ and sediment deposition $D(x, t)$ are fixed at zero. However if the water depth is less than $\mathcal{H}_0$, then the unit discharge is also fixed at zero, see Figure 2.8. This interval improves the numerical stability, but it can induce a reduction of the zones where the wet-dry front and the bed evolution are approximated. In [19] is proposed that only values of the characteristic speeds of wet cells are taken into account. This criteria is written as follows.

$$\lambda_i = \begin{cases} u_i - \sqrt{\frac{gw_{1,i}}{2}} & \text{if } w_{1,j} < \mathcal{H}_1, \\ \lambda_i & \text{else.} \end{cases} \quad (2.87)$$

$$\lambda_{i+1} = \begin{cases} u_i + 2\sqrt{\frac{gw_{1,i}}{2}} & \text{if } w_{1,j} < \mathcal{H}_1, \\ \lambda_{i+1} & \text{else.} \end{cases} \quad (2.88)$$

2.5. Numerical Results

In order to measure the accuracy of the numerical schemes described earlier, several numerical tests were carried out. In the first test, the steady stationary solution conditions are recreated. In the second test, a dam break, when the whole spatial domain is wet, is reproduced. Finally, three experimental dam breaks were reproduced and the numerical results were compared with the experimental data.

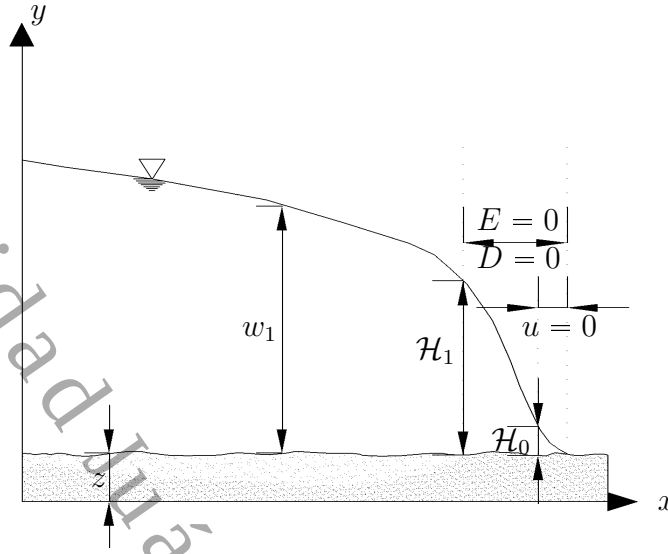


Figura 2.8: Interval to treat the bed evolution close of wetting drying fronts when the HLLCS approximate Riemann solver is used.

2.5.1. Parameters for both the mathematical model and the numerical model, initial conditions and boundary conditions

The values for the physical parameters are shown in Table 2.2, while the values for the numerical parameters, for each numerical scheme, are displayed in Tables 2.3 and 2.4. In these tables we can see different values, for the same parameters, for the Q-schemes and HLLCS scheme. These differences come from the fact that these parameters have been fitted, by repeating several times the numerical tests, in order to reach the best results for every experiment. The differences between the numbers of nodes used by each schemes, are due to the HLLCS scheme needs more nodes to reach the machine precision for the computation of the velocity when the steady solution is computed. The wet-dry parameter has been calibrated numerically for each test, a study about the influences of that parameter on the results of shallow water flow simulation has been reported in [22]. The values for the dimensionless parameters are shown in Table 2.5. The numerical tests mentioned in these tables are as follows:

- Test 1. Computation of steady stationary solution. In this test the domain is $\Omega = [-5, 10]$. The initial conditions are given by:

$$z(x, t_0) = \begin{cases} \sin(x + 6) + 1 & \text{if } \frac{3\pi}{2} - 6 \leq x \leq \frac{7\pi}{2} - 6, \\ 0 & \text{else.} \end{cases}$$

$$w_1(x, t_0) = \begin{cases} 0 & \text{if } \frac{3\pi}{2} - 6 \leq x \leq \frac{7\pi}{2} - 6, \\ 1.5 - z(x, t_0) & \text{else.} \end{cases}$$

$$u(x, t_0) = 0, \quad \text{and} \quad c(x, t_0) = \begin{cases} 0 & \text{if } \frac{3\pi}{2} - 6 \leq x \leq \frac{7\pi}{2} - 6, \\ 0.3 & \text{else.} \end{cases}$$

A wall boundary condition is considered on the left and right end of the domain.

- Test 2. Simulation of a dam break, described in [8]. The domain in this test is $\Omega = [20000, 30000]$. The initial conditions are given by:

$$z(x, t_0) = 0, \quad w_1(x, t_0) = \begin{cases} 40 & \text{if } 20000 \leq x \leq 25000, \\ 2 & \text{else.} \end{cases}$$

$$u(x, t_0) = 0, \quad \text{and} \quad c(x, t_0) = 0.$$

Open boundary conditions are imposed at both ends of the domain.

- Test 3. Simulation of an experimental dam break carried out at the Université Catholique of Louvain, and shown in [42]. The domain is $\Omega = [0, 2.5]$. The initial conditions read

$$z(x, t_0) = 0, \quad w_1(x, t_0) = \begin{cases} 0.1 & \text{if } 0 \leq x \leq 1.25, \\ 0 & \text{else.} \end{cases}$$

$$u(x, t_0) = 0, \quad \text{and} \quad c(x, t_0) = 0.$$

Open boundary conditions are imposed at both ends of the domain.

- Test 4 and 5. Numerical simulation of conditions for a small-scale laboratory dam break, as in [38]. In both cases, the domain is $\Omega = [-3.0, 3.0]$. The initial conditions of the fourth test are given by the following relations:

$$z(x, t_0) = 0, \quad w_1(x, t_0) = \begin{cases} 0.35 & \text{if } -3 \leq x \leq 0, \\ 0 & \text{else.} \end{cases}$$

$$u(x, t_0) = 0, \quad \text{and} \quad c(x, t_0) = 0.$$

The initial conditions of the fifth test read

$$z(x, t_0) = \begin{cases} 0.1 & \text{if } -3 \leq x \leq 0, \\ 0 & \text{else.} \end{cases}, \quad w_1(x, t_0) = \begin{cases} 0.25 & \text{if } -3 \leq x \leq 0, \\ 0.1 & \text{else.} \end{cases}$$

$$u(x, t_0) = 0, \quad \text{and} \quad c(x, t_0) = 0.$$

In both tests open boundary conditions, at both ends of the domain, are imposed.

Remark 1 The numerical scheme, where the source terms $\mathbf{S}_1$ and $\mathbf{S}_4$ are discretised in upwind form, is denoted by QUS, while the numerical scheme which implements the non-conservative system is denoted by QNC.

Tabla 2.2: Physical parameters for numerical tests.

Test	ρ_s (kg/m ³)	ρ_w (kg/m ³)	s	d (mm)	ν (m ² /s)
Test 1	2650	1000	1.650	4	1.2×10^{-6}
Test 2	2650	1000	1.650	4	1.2×10^{-6}
Test 3	1540	1000	1.540	3.5	1.2×10^{-6}
Test 4	2580	1000	1.580	3.9	1.2×10^{-6}
Test 5	2580	1000	1.580	3.9	1.2×10^{-6}

Tabla 2.3: Numerical parameters for QUS and QNC, $\hat{t}_0 = \sqrt{h_l/g}$.

Parameters	CFL	ε_{HR}	ε_{WD}	ε_{ED}	T_f (s)	Nodes	A_g
Test 1	0.5	0.0001	0.0001	0.0012	60	300	0.01
Test 2	0.7	10	0.001	0.001	120	1000	0
Test 3	0.3	3	0.005	0.001	$10 \hat{t}_0$	300	0.008
Test 4	0.35	5	0.003	0.01	1.25	600	0.0025
Test 5	0.35	5	0.008	0.01	1.25	600	0.0065

2.5.2. Test 1

In Figures 2.9-2.12, we can see the numerical results given by three numerical schemes. Figure 2.9 shows the comparison between the initial free surface and the final computed free surface. From this figure we can see no variations of the final free surface with respect of the initial free surface. This fact is confirmed in Figure 2.10, where the error of the computed free surface is shown. Notice that the errors of the three numerical schemes are in the order of machine epsilon. As well, in Figure 2.9 it is observed that the depth of the bottom increases as an effect of the sediment deposition, as it was expected. Figure 2.11 shows that the three numerical schemes compute null velocity over the whole domain, because the computed velocities are in the order of machine epsilon. From equation (2.74) we have that the volumetric sediment concentration decreases as a function of time, if $c(x, t) < 1 - p$. This fact is well recreated by our numerical scheme, as it is depicted in Figure 2.12, where the absolute value of the difference between the

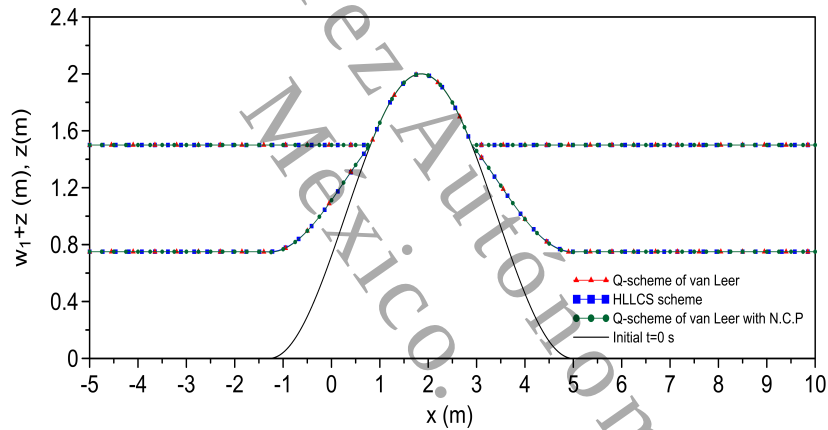
Tabla 2.4: Numerical parameters for HLLCS approximate Riemann solver, $\hat{t}_0 = \sqrt{h_l/g}$.

Parameters	CFL	$(\mathcal{H}_0, \mathcal{H}_1)$	T_f (s)	Nodes	A_g
Test 1	0.4	(0.01, 0.012)	60	5000	0.01
Test 2	0.4	(0, 0.001)	120	1000	0
Test 3	0.4	(0.05, 0.07)	$10 \hat{t}_0$	150	0.0011
Test 4	0.4	(0, 0.001)	1.25	600	0.0011
Test 5	0.4	(0, 0.001)	1.25	400	0.005

Tabla 2.5: Dimensionless values used in numerical tests.

Parameters	η		θ_c		ϕ		p	
	QS	HLLCS	QS	HLLCS	QS	HLLCS	QS	HLLCS
Test 1	0.02	0.02	0.045	0.045	10	10	0.4	0.4
Test 2	0.03	0.02	0.045	0.045	10	10	0.4	0.4
Test 3	0.03	0.02	0.045	0.045	4	0.4	0.4	0.4
Test 4	0.0165	0.009	0.047	0.047	2	0.5	0.47	0.47
Test 5	0.0165	0.009	0.047	0.047	1	0.2	0.47	0.47

left hand side and the right hand side of (2.74) can be seen. Notice that the numerical schemes are computing a difference in the order of 1×10^{-15} . Therefore we can conclude that the numerical schemes presented in this work compute correctly the steady solution, i.e. that our numerical schemes are well balanced.

Figura 2.9: Free surface of water and depth of bed at $t = 60$ s.

2.5.3. Test 2

The dam break conditions shown in [8] were simulated in this numerical test and our numerical results are compared with those provided in that reference. Both the water free surface and the depth of bed at $t = 20$ s are shown in Figure 2.13. Note that the water free surface computed by the three numerical schemes is an accurate approximation of the results shown in [8]. Regarding the numerical results of the depth of bed, it can be seen that the HLLCS scheme estimates more sediment entrainment where the dam break was centered, while the schemes QUS and QNC accurately approximate the numerical results shown in [8].

Figures 2.14 and 2.15 show the results obtained for the velocity at $t = 20$ s and $t = 120$ s, respectively. The results provided by the three numerical schemes at $t = 20$ s

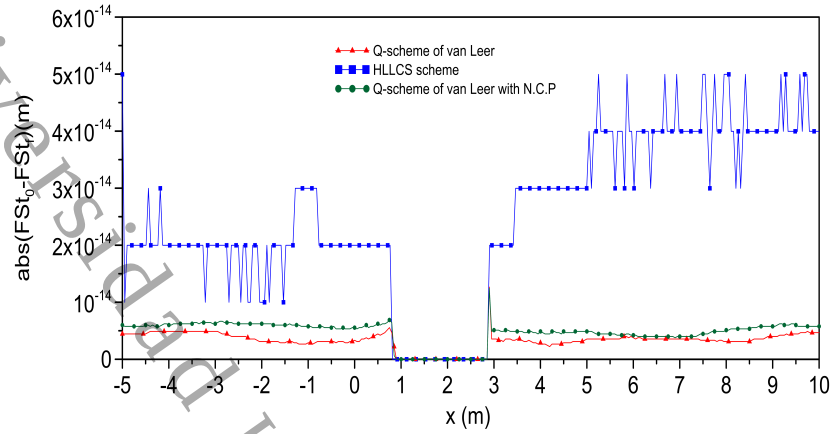


Figura 2.10: Error in the computation of the free surface of water at $t = 60$ s.

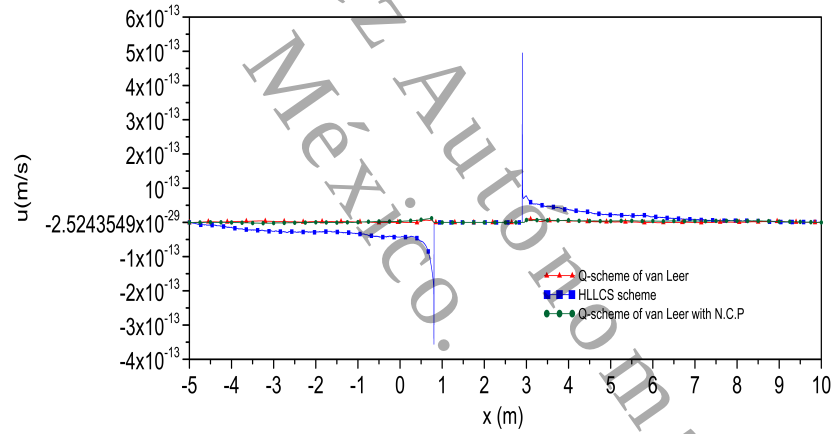


Figura 2.11: Velocity at $t = 60$ s.

are very similar to the results of Cao. However, for $t = 120$ s in the region where the water front is computed, the velocity is less than the one of [8] and the position of the front is computed ahead of the front showed by Cao, but these differences are due to the way the source terms are numerically treated, another possible error source of these difference may be the way the data were obtained.

Both the expansion wave and shock wave shown by Cao are recreated exactly by the three numerical schemes. Likewise, the velocity and morphological changes are estimated properly. Thus, the three numerical schemes reproduce good enough the numerical results of [8].

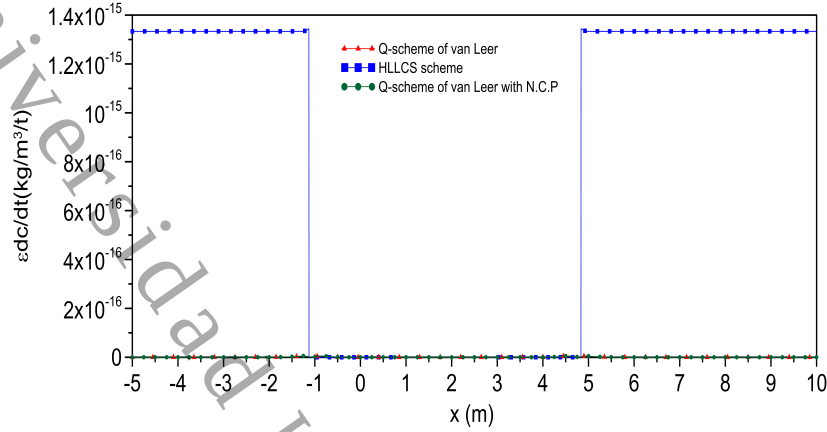


Figura 2.12: Error for the computation of the temporal evolution of the volumetric sediment concentration at $t = 60$ s.

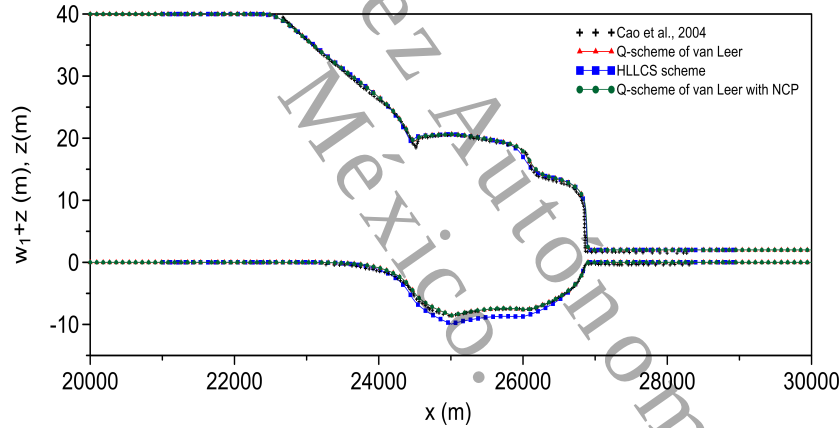


Figura 2.13: Water free surface and bed the depth computed at $t = 120$ s.

2.5.4. Test 3

In order to measure the accuracy of our numerical schemes, the experimental dam break flows carried out in Louvain-la-Neuve (Université Catholique de Louvain [20]) were numerically simulated and the results obtained were compared with the experimental data.

Figures 2.16 – 2.18 show the numerical results of the three numerical schemes against the experimental data. The downstream free surface of water computed by schemes QUS and QNC is greater than the water free surface given for the experimental data in this region. In these figures, we can see that QUS and QNC numerical schemes compute the position of the front with high degree of precision for every computed time. Moreover, the results provided by HLLCS scheme are an accurate approximation for the downstream water free surface and the position of the front is computed with high

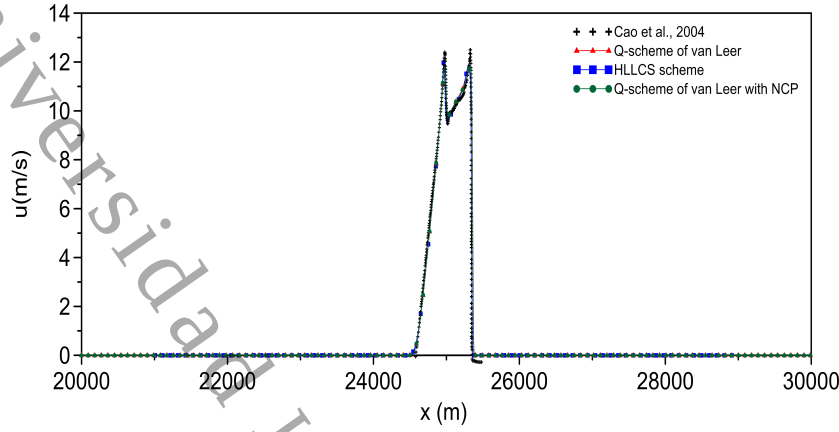


Figura 2.14: Velocity computed at $t = 20$ s.

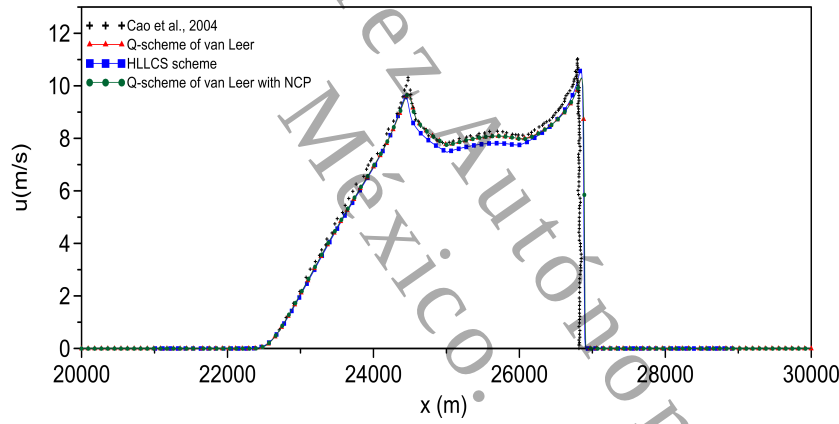


Figura 2.15: Velocity computed at $t = 120$ s.

degree of precision as well. It should be noted that the obtained up-stream numerical water free surface is less than the experimental data of the water free surface in this region, but the differences are minor. The hydraulic jump observed in the experimental data is better approximated by the HLLCS scheme. This allows us to conclude that the three numerical schemes compute in a satisfactory way the free surface.

Regarding the bed evolution, Figures 2.16 and 2.17, show that the up-stream morphological evolution computed by the HLLCS scheme, at times $5\hat{t}_0$ and $7.5\hat{t}_0$, are less than the morphological changes observed in the experimental data. On the other hand, the upstream morphological evolution computed by both schemes QUS and QNC is in accordance with the experimental data. Furthermore at time $10\hat{t}_0$ both schemes QUS and QNC compute more upstream sediment entrainment than that observed in the experimental data, while the upstream sediment entrainment computed by the HLLCS scheme fits better the experimental data. The downstream sediment entrainment com-

puted by the three numerical schemes is greater than the experimental data for every computed time, although the computed approximation is good enough.

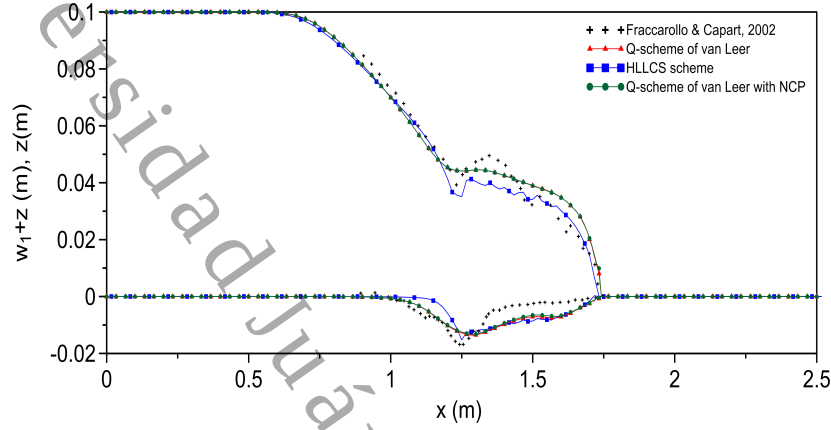


Figura 2.16: Water free surface and the bed depth computed at $5\hat{t}_0$.

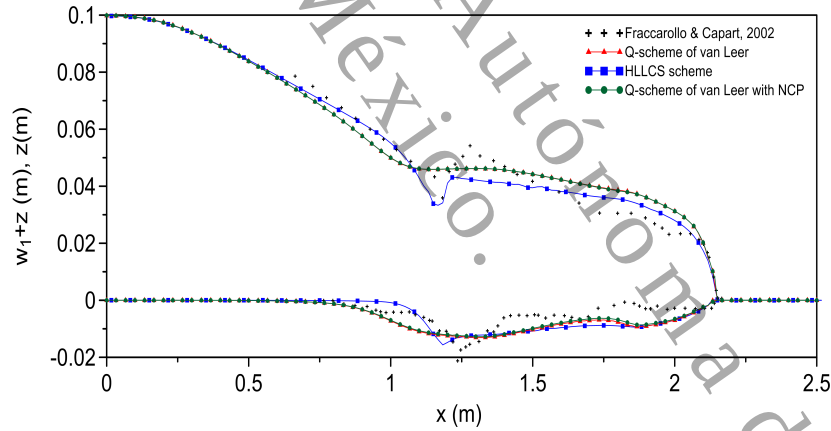


Figura 2.17: Water free surface and the bed depth computed at $7.5\hat{t}_0$.

2.5.5. Tests 4 and 5

In these numerical tests the conditions of small scale dam breaks, performed in the Université Catholique of Louvain, were recreated. Both the values for physical parameter and the experimental data were obtained from [38].

The results computed in the fourth numerical test are shown in Figure 2.19.

It can be seen that the downstream sediment entrainment computed by the HLLCS scheme is greater than the downstream sediment entrainment provided by the experimental data. However, the differences between them are small. The upstream sediment

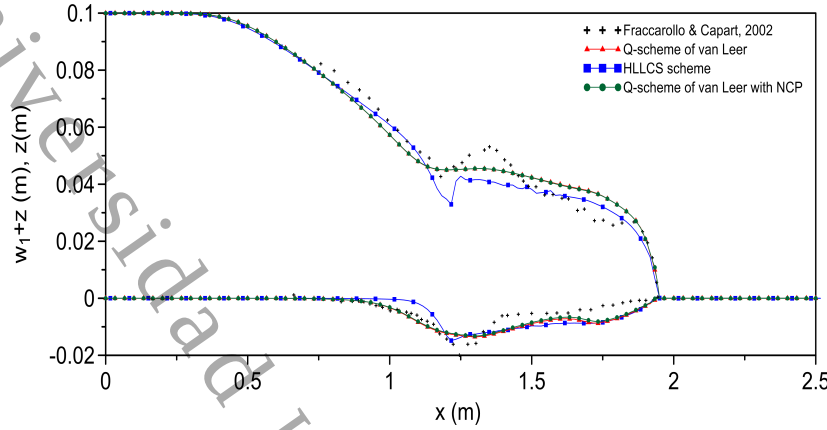


Figura 2.18: Water free surface and the bed depth computed at $10 \hat{t}_0$.

entrainment computed by the HLLCS scheme is in accordance with the experimental data. While the bed evolution computed by both schemes QUS and QNC reproduces accurately the bed changes.

From Figure 2.19 it can be seen that the region where the hydraulic jump took place is well computed by the three numerical schemes, but the experimental water free surface, in that area, is not given correctly by the numerical results.

The downstream water free surface is correctly reproduced by the numerical schemes because the numerical results and experimental data have the same qualitative form. Notice that the downstream free surface of water computed by both schemes QUS and QNC is greater than the experimental water free surface although the adjustment made is adequate. Moreover, the water free surface computed by the HLLCS scheme fits accurately the experimental downstream water free surface.

Regarding the water front advance, it can be seen that the results obtained by the HLLCS scheme are ahead of the position provided by the experimental data while the results provided by the QUS and QNC methods computed accurately that position.

The numerical results of the fifth numerical test are shown in Figure 2.20. We can see that the hydraulic jump is only partially reproduced by the three numerical schemes because there is a difference between the position provided by the experimental data and the calculated one by the numerical schemes. The computed water free surface after the hydraulic jump is in agreement with the experimental data. Notice that the water front advance computed by the three numerical schemes presents a gap in comparison with the water front advance observed in the experimental data. Regarding the morphological changes, the tendency observed in the experimental data is correctly represented by the three numerical schemes. The HLLCS scheme produces more upstream sediment entrainment than the one observed in the experimental data, but the differences are minor. The adjustment of the upstream bed evolution made by the schemes QUS and QNC is satisfactory. The downstream bed evolution computed by the three numerical

schemes is in good agreement with the experimental bed evolution. In these numerical tests small oscillations, produced by the HLLCS scheme, are noticeable where the hydraulic jump took place, while the results of QUS and QNC give a smooth tendency.

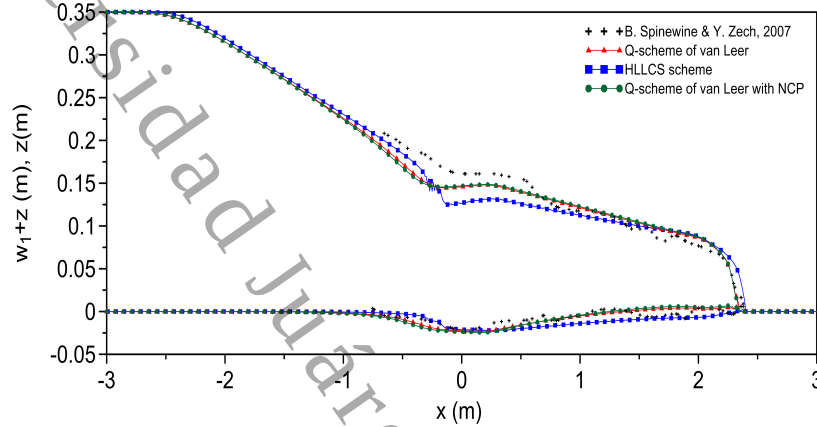


Figura 2.19: Free surface of water and bed depth computed at $t = 1.25$ s.

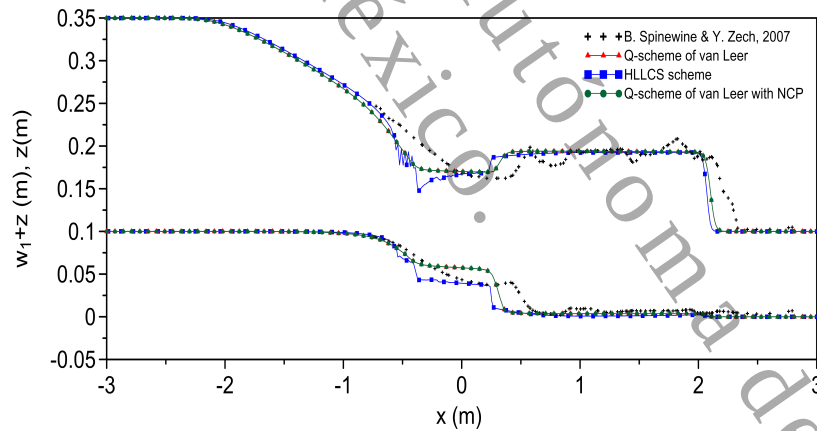


Figura 2.20: Free surface of water and bed depth computed at $t = 1.25$ s.

2.6. Conclusions

Three numerical schemes to simulate dam break flows with suspended load and bedload sediment transport have been presented. Two of them implement the Q-scheme of van Leer to compute the numerical flux and the other one implements a HLLC Riemann solver. In the named QUS scheme, both the bed slope source term and the source term with the sediment concentration variations are discretised in an upwind

way. In the named QNC scheme, the bed slope source term is discretised in an upwind way but the source term related to the variation of sediment concentration is dealt as a non conservative product. While, in the HLLCS Riemann solver, the bed slope source term is taken into account inside of the numerical flux, the sediment concentration variation is computed in a explicit way. We have proved that the numerical schemes are well-balanced, by the numerical results of the steady stationary solution test. The contrast made between the numerical results and the experimental data shows that the numerical schemes accurately compute the water front advance. It can be seen that the computed free surface of water successfully recreated the free surface of water of the experimental data. Regarding the bed evolution, we have seen that the three numerical schemes compute a successful approximation for the observed bed evolution in the experimental data. As the used Grass parameters are close to zero, the interaction between the flux and the bottom is weak for every physical experiments simulated in this work.

From the several simulations done for each experiment, we observed that both the constant value ϕ , which appears in the formulation for sediment entrainment, and the wet-dry parameter ε_{WD} , influence the computations of the water front. Regarding the sediment entrainment, the threshold value ε_{ED} plays an important role for Q-schemes and the interval $(\mathcal{H}_0, \mathcal{H}_1)$ for HLLCS scheme, because it helps to avoid the unphysical values for sediment entrainment which could cause numerical instabilities. Therefore, carry out an uncertainty quantification study of these parameters is an interesting future work, as it shall allow us to determine optimum values for each parameter and measure how much the variability of these parameters influences the quality of the results. Other future work is the bidimensional extension of the model and of the numerical schemes. The bidimensional extension of the mathematical model is effortless to do and the numerical scheme can be built solving projected Riemann problems on the edge of the bidimensional finite volume mesh, i.e. the bidimensional problem is computed using the one dimensional flux applied to the projected problem. We consider as field of the applications for this work the study of settlement in river, the study of the settlement in sewage treatment plants, to study the morphological changes on river bottoms and how this phenomenon affects the flux, as well, the measurement of the impact of river overflowing.

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Capítulo 3

Conclusiones y Recomendaciones

En este capítulo se hace una discusión más extensa de las conclusiones hechas en el artículo de investigación, así como también, se mencionan posibles futuras líneas de investigación en las que se pueden desarrollar para ayudar a la comprensión y resolución del problema del transporte de sedimentos.

3.1. Conclusiones

En el presente trabajo se desarrollaron tres esquemas numéricos, dos de ellos se basan en la construcción de flujo numérico mediante el Q-esquema de Van-leer, en el cual, la matriz de viscosidad numérica se toma en cuenta mediante la matriz valor absoluto de la jacobiana del flujo físico, dicha matriz es evaluada en la media aritmética de los estados a la derecha e izquierda de la frontera común de dos celdas de volumen finito. El tercer esquema, se basa en las ideas del flujo numérico HLLC, al cual se agrega una cuarta onda en la deducción del flujo numérico, la cual se corresponde con el término fuente que tiene en cuenta a los procesos de erosión y depósito.

La diferencia entre los esquemas numéricos basados en el Q-esquema de Van-leer es la forma en cómo se trata la discretización del término fuente, en una primera, tanto el término fuente de la pendiente del fondo, como el término fuente con la erosión y el depósito son discretizados de forma descentrada, mientras que en la segunda forma, solo el término fuente de la pendiente del fondo se discretiza de forma descentrada y el término fuente de la erosión y depósito se considera como una parte no conservativa, lo cual origina que se considere como una parte del flujo físico que se utiliza en el esquema numérico. Por su parte, el esquema tipo HLLC-S toma en cuenta los saltos del término fuente asociado a la erosión y el depósito.

Se demostró analíticamente que los esquemas tipo Q-esquema verifican la propiedad de conservación, es decir, pueden calcular de forma exacta o en el orden de precisión de la máquina, la solución del agua en calma para el modelo que se está resolviendo. Adicionalmente, se aportó evidencia numérica robusta (a precisión de máquina) que sugiere que el esquema HLLCS exhibe este mismo comportamiento conservativo.

Numéricamente se ha mostrado que los Q-esquemas dan la solución aproximada exacta, mientras que el esquema tipo HLLC-S, presenta algunos de problema en la zonas cercanas donde se dan las apariciones de frentes de seco-mojado.

Los esquemas numéricos dan buenos resultados en lo casos en que se contrastan con los datos obtenidos de laboratorios. El frente de avance del agua se calcula de buena forma, así como también la evolución del fondo sedimentado. Las soluciones numéricas no presentan principios de inestabilidades, se calculan soluciones suaves. Cabe mencionar que, la superficie libre del agua aproximada por los esquemas numéricos, sobre estima aguas abajo la superficie libre del agua proporcionada por los datos experimentales, este fenómeno se puede observar en cada uno de los experimentos realizados.

Se pueden observar, de los resultados numéricos, que el tener en cuenta el transporte de sedimentos da origen a incrementos en la velocidad, así como a zonas donde se tienen resaltos hidráulicos, por lo cual, para tener un panorama más realista de lo que una inundación puede ocasionar es importante tener en cuenta los fenómenos del transporte de sedimentos. Más aún, para hacer más realista el modelo, se podría considerar a la presión hidrodinámica.

De los diversos experimentos llevados a cabo, un factor que es importante al momento de calcular el frente de avance del agua, es el parámetro de tolerancia empleado en los frentes de seco mojado. Valores muy pequeños de este parámetro, pueden ocasionar que hayan inestabilidades numéricas en el esquema, mientras que valores un poco mayores del mismo, pueden originar la aparición de ondas de choques, las cuales, teóricamente no se dan en los frentes de seco-mojado, y con ello retrasar el frente de avance del agua. Otro parámetro numérico que tiene importancia en la calidad de la solución numérica es el parámetro de regularización, utilizado en la regularización parabólica de Harte, el cual se usa para evitar que se pierda la información asociada a alguno de los valores propios, cuando se pasa por un punto sónico. Valores pequeños de este parámetro pueden ocasionar inestabilidades numéricas por la anulación de algún valor propio, y valores muy grandes del mismo pueden ocasionar soluciones erróneas por dar demasiada viscosidad numérica al esquema.

3.2. Trabajos futuros

En la linea de investigación del transporte de sedimentos y para dar continuidad con el trabajo realizado en esta tesis, se podrían explorar las ideas siguiente.

1. Implementar los mismos esquemas numéricos con las diferentes leyes del transporte de fondo, para medir el impacto que estos tienen en la predicción de la evolución morfológica.
2. Hacer la extensión a dimensión dos del modelo matemático y la respectiva extensión del método numérico trabajado.

3. Acoplar la solución numérica para que el transporte de fondo se calcule al mismo tiempo que la carga en suspensión, lo cual implicará definir un esquema numérico que pueda capturar las características de dicho fenómeno.
4. Considerar a la densidad de la mezcla del agua y sedimento como parte de las variables conservadas y solucionar con el esquema numérico para dichas variables, a partir de ello analizar si hay alguna mejora en la calidad de las soluciones.
5. Deducir modelos multi capas para el transporte de sedimentos, para poder tomar en cuenta la estratificación del fenómeno conforme se va dando la interacción del sedimento con el agua. Plantear esquemas numéricos bien equilibrados para la solución de los modelos multicapas.
6. Extender los esquemas numéricos del transporte de sedimentos a esquemas numéricos de orden superior en espacio y buscar al menos el segundo orden en tiempo.
7. Hacer análisis de cuantificación de la incertidumbre que los parámetros meten al modelo matemático y a partir de ello determinar el mejor conjunto de parámetros para las situaciones que se busquen recrear con los modelos trabajados.

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Autor(a) o autores(ras) de la Tesis:	Juan Carlos González Aguirre
ORCID:	0000-0002-7067-5527
Resumen de la Tesis:	El transporte de sedimentos puede ser modelado usando modelos hidrodinámicos basados en las ecuaciones de las aguas someras acopladas con la ecuación de conservación del sedimento y la ecuación de conservación del fondo sedimentado. El sistema se complementa con las ecuaciones de la conservación de energía y la ecuación de Exner para el transporte por carga de fondo. La solución numérica del sistema completo se hace de forma segregada, primero se resuelve el sistema hiperbólico de leyes de balance usando un esquema de volúmenes finitos. Para este paso, en este trabajo se han considerado tres formas de calcular el flujo numérico, las cuales son, el Q-esquema de van Leer, el flujo HLLCS, y el tercero donde se toma en cuenta la derivada espacial de la concentración de sedimentos en un término no conservativo. La discretización del término fuente se lleva a cabo de acuerdo al flujo numérico elegido. En la segunda etapa, se resuelve la ecuación de la conservación del fondo sedimentado utilizando la información calculada en la primera etapa, para la determinación de una velocidad del sedimento.
Palabras claves de la Tesis:	Sedimento, Flujo, Volúmenes finitos, transporte suspendido, transporte por carga de fondo
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